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A geometric Birkhoffian formalism for nonlinear RLC networks

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Abstract

The aim of this paper is to give a formulation of the dynamics of nonlinear RLC circuits as a geometric Birkhoffian system and to discuss in this context the concepts of regularity, conservativeness, dissipativeness. An RLC circuit, with no assumptions placed on its topology, will be described by a family of Birkhoffian systems, parameterized by a finite number of real constants which correspond to initial values of certain state variables of the circuit. The configuration space and a special Pfaffian form, called the Birkhoffian, are obtained from the constitutive relations of the resistors, inductors and capacitors involved and from Kirchhoff's laws. Under certain assumptions on the voltage–current characteristic for resistors, it is shown that a Birkhoffian system associated with an RLC circuit is dissipative. For RLC networks which contain a number of pure capacitor loops or pure resistor loops the Birkhoffian associated is never regular. A procedure for reducing the original configuration space to a lower dimensional one, thereby regularizing the Birkhoffian, it is also presented. In order to illustrate the results, specific examples are discussed in detail.

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1. Introduction

Lagrangian and Hamiltonian mechanics continue to attract a large amount of attention in the literature, because many mechanical and electromechanical systems may be modelled within

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these frameworks. During the past ten years, a far-reaching generalization of the Hamiltonian framework has been developed in a series of papers. This generalization, which is based on the geometric notion of generalized Dirac structure, gives rise to implicit Hamiltonian systems (see for example papers by Maschke and van der Schaft [10], van der Schaft [12]). In the papers by Yoshimura and Marsden [15] the concept of Dirac structures and the variational principle are used to define and develop the basic properties of implicit Lagrangian systems.

An alternative approach to the study of dynamical systems which appears to cover a wide class of systems, among them the nonholonomic systems, the degenerate systems and the dissipative ones, is the Birkhoffian formalism, a global formalism of the dynamics of implicit systems of second-order ordinary differential equations on a manifold. The classical book by Birkhoff [2] contains in Chapter I many interesting ideas about classical dynamics from the viewpoint of differential geometry. In order to present these ideas in a coordinate free fashion, one considers the formalism of 2-jets (see for example Kobayashi and Oliva [7]). The space of configurations is a smooth m-dimensional differentiable connected manifold and the covariant character of the Birkhoff generalized forces is obtained by introducing the notion of elementary work, called the Birkhoffian, a special Pfaffian form defined on the 2-jets manifold. The dynamical system associated with this Pfaffian form is a subset of the 2-jets manifold which defines an implicit second-order ordinary differential system. The notion of the Birkhoffian allows one to formulate the concepts of reciprocity, regularity, affine structure in the accelerations, conservativeness, in an intrinsic way.

The electrical circuits theory benefits from many tools developed in mathematics. In order to study the dynamics of LC and RLC electrical circuits, various Lagrangian and Hamiltonian formulations have been considered in the literature (see for example [3,4,9,10,12,13], and the references therein). To describe LC circuit equations, Hamiltonian formulations have been used more often. In [9], the dynamics of a nonlinear LC circuit is shown to be of Hamiltonian nature with respect to a certain Poisson bracket which may be degenerate, that is, nonsymplectic. The dynamics of "complete" RLC networks was described by Brayton and Moser [4] in terms of a function of inductor currents and capacitor voltages, called the mixed potential function. However, for all those formulations, a certain topological assumption on the electrical circuit appears to be crucial, that is, the circuit is supposed to contain neither loops of capacitors nor cutsets of inductors. In [10,12] and [3] the Poisson bracket is replaced by the more general notion of a Dirac structure on a vector space, leading to implicit Hamiltonian systems. In this formalism, it is possible to include LC networks which do not obey the topological assumption mentioned before. In [15] an example of an LC circuit in the context of implicit Lagrangian systems is given for a degenerate Lagrangian system with holonomic constraints.

The potential relevance of the Birkhoffian formalism in the context of electrical circuits is discussed by Ionescu and Scheurle [8], where a formulation of general nonlinear LC circuits within the framework of Birkhoffian dynamical systems on manifolds is presented. In [8] specific examples of electrical networks are discussed in this framework. These are networks which contain closed loops formed by capacitors, as well as inductor cutsets, and also LC networks which contain independent voltage sources as well as independent current sources.

In the paper at hand we present a formulation of the dynamics of nonlinear RLC electrical circuits within the framework of Birkhoffian systems. On the basis of Kirchhoff's laws and the constitutive relations for the resistors, inductors, capacitors involved, we get for a nonlinear RLC circuit, a whole family of configuration spaces and special Pfaffian forms, called Birkhoffians. The configuration spaces are parameterized by a finite number of real constants which correspond to initial values of certain state variables of the circuit. In particular, we can allow pure capacitor loops as well as pure inductor cutsets. Under certain assumptions on the voltage–current characteristic for resistors, it is shown that a Birkhoffian system associated with an RLC circuit is dissipative.

The paper is organized as follows. In Section 2, we recall the basics of Birkhoffian systems (see [2]) presented from the viewpoint of differential geometry using the formalism of jets (see [7]). In particular, we introduce the notion of a dissipative Birkhoffian system, in order to be able to treat the case of RLC networks in the next section. In Section 3, our Birkhoffian formulation of the dynamic equations of a nonlinear RLC circuit is given. Properties of the corresponding Birkhoffian such as its regularity and its dissipativeness are also discussed in this section. For electrical RLC networks which contain a number of only capacitor loops or only resistor loops, we present a systematic procedure for reducing the original configuration space to a lower dimensional one, thereby regularizing the Birkhoffian. On the reduced configuration space the reduced Birkhoffian will still be dissipative. Finally, in Section 4 we consider two specific examples. These examples are intended to serve our purpose of demonstrating the power of the Birkhoffian approach.

2. Birkhoffian systems

For a smooth m-dimensional differentiable connected manifold M, we consider the tangent bundles (TM, π_M, M) and (TTM, π_{TM}, TM). Let $q = (q^1, q^2, \ldots, q^m)$ be a local coordinate system on M. This induces natural local coordinate systems on TM and TTM, denoted by $(q, \dot{q})$, respectively $(q, \dot{q}, \mathrm{d}q, \mathrm{d}\dot{q})$. The *2-jets manifold* $J^2(M)$ is a $3m$-dimensional submanifold of TTM defined by

$$J^2(M) = \{z \in TTM \mid T\pi_M(z) = \pi_{TM}(z)\} \tag{2.1}$$

where $T\pi_M : TTM \to TM$ is the tangent map of π_M. We write $\pi_J := \pi_{TM}|_{J^2(M)} = T\pi_M|_{J^2(M)}$. $(J^2(M), \pi_J, TM)$, called the *2-jet bundle* (see [7]), is an affine bundle modelled on the vertical vector bundle $(V(M), \pi_{TM}|_{V(M)}, TM)$, $V(M) = \bigcup_{v\in TM} V_v(M)$, where $V_v(M) = \{z \in T_vTM | (T\pi_M)_v(z) = 0\}$. In [1,11] this bundle is denoted by $T^2(M)$ and called a *second-order tangent bundle*. In natural local coordinates, the equality in (2.1) yields $(q, \dot{q}, \dot{q}, \mathrm{d}\dot{q}|_{J^2(M)})$ as a local coordinate system on $J^2(M)$. We set $\ddot{q} := \mathrm{d}\dot{q}|_{J^2(M)}$. Thus, a local coordinate system q on M induces the natural local coordinate system $(q, \dot{q}, \ddot{q})$ on $J^2(M)$. For further details on this affine bundle see [1,7,11].

A *Birkhoffian* corresponding to the configuration manifold M is a smooth 1-form ω on $J^2(M)$ such that, for any $x \in M$, we have

$$i_x^*\omega = 0 \tag{2.2}$$

where $i_x : \beta^{-1}(x) \to J^2(M)$ is the embedding of the submanifold $\beta^{-1}(x)$ into $J^2(M)$, $\beta = \pi_M \circ \pi_J$. From this definition it follows that, in the natural local coordinate system $(q, \dot{q}, \ddot{q})$ of $J^2(M)$, a Birkhoffian ω is given by

$$\omega = \sum_{j=1}^{m} Q_j(q, \dot{q}, \ddot{q})\mathrm{d}q^j \tag{2.3}$$

with certain functions $Q_j : J^2(M) \to \mathbf{R}$.
The pair (M, ω) is said to be a *Birkhoff system* (see [7]).

The *differential system associated with a Birkhoffian* ω (see [7]) is the set (maybe empty) $D(\omega)$, given by

$$D(\omega) := \left\{ z \in J^2(M) | \omega(z) = 0 \right\}. \tag{2.4}$$

The manifold M is the *space of configurations* of $D(\omega)$, and $D(\omega)$ is said to have m 'degrees of freedom'. The Q_i are the 'generalized external forces' associated with the local coordinate system (q). In the natural local coordinate system, $D(\omega)$ is characterized by the following implicit system of second-order ODE's

$$Q_j(q, \dot{q}, \ddot{q}) = 0 \quad \text{for all } j = \overline{1, m}. \tag{2.5}$$

We conclude that the Birkhoffian formalism is a global formalism for the dynamics of implicit systems of second-order differential equations on a manifold.

Let us now associate a vector field with a Birkhoffian ω.

A *vector field* Y on the manifold TM is a smooth function $Y : TM \to TTM$ such that $\pi_{TM} \circ Y = \text{id}$. Any vector field Y on TM is called a *second-order vector field on TM* if and only if $T\pi_M(Y_v) = v$ for all $v \in TM$.

A cross section X of the affine bundle $(J^2(M), \pi_J, TM)$, that is, a smooth function $X : TM \to J^2(M)$ such that $\pi_J \circ X = \text{id}$, can be identified with a special vector field on TM, namely, the second-order vector field on TM associated with X. Indeed, because $(J^2(M), \pi_J, TM)$ is a sub-bundle of (TTM, π_{TM}, TM) as well as of $(TTM, T\pi_M, TM)$, its sections can be regarded as sections of these two tangent bundles. Thus, using the canonical embedding $i : J^2(M) \to TTM$, X can be identified with Y, that is, $Y = i \circ X$.

In natural local coordinates, a second-order vector field can be represented as

$$Y = \sum_{j=1}^{m} \left[\dot{q}^j \frac{\partial}{\partial q^j} + \ddot{q}^j(q, \dot{q}) \frac{\partial}{\partial \dot{q}^j} \right]. \tag{2.6}$$

A *Birkhoffian vector field* associated with a Birkhoffian ω of M (see [7]) is a smooth second-order vector field on TM, $Y = i \circ X$, with $X : TM \to J^2(M)$, such that $\text{Im}\, X \subset D(\omega)$, that is

$$X^*\omega = 0. \tag{2.7}$$

In the natural local coordinate system, a Birkhoffian vector field is given by the expression (2.6), such that $Q_j(q, \dot{q}, \ddot{q}(q, \dot{q})) = 0$.

A Birkhoffian ω is *regular* if and only if

$$\det \left[\frac{\partial Q_j}{\partial \ddot{q}^i}(q, \dot{q}, \ddot{q}) \right]_{i,j=1,\ldots,m} \neq 0 \tag{2.8}$$

for all $(q, \dot{q}, \ddot{q})$, and for each $(q, \dot{q})$, there exists $\ddot{q}$ such that $Q_j(q, \dot{q}, \ddot{q}) = 0$, $j = 1, \ldots, m$.

If a Birkhoffian ω of M is regular, then it satisfies *the principle of determinism*, that is, there exists a unique Birkhoffian vector field $Y = i \circ X$ associated with ω such that $\text{Im}\, X = D(\omega)$ (see [7]).

A Birkhoffian ω of the configuration space M is called *conservative* if and only if there exists a smooth function $E_\omega : TM \to \mathbf{R}$ such that

$$(X^*\omega)Y = \mathrm{d}E_\omega(Y) \tag{2.9}$$

for all second-order vector fields $Y = i \circ X$ (see [7]).

Eq. (2.9) is equivalent, in the natural local coordinate system, to the identity (see [2], p. 16, Eq. (4))

$$\sum_{j=1}^{m} Q_j(q\,\dot{q},\ddot{q})\dot{q}^j = \sum_{j=1}^{m}\left[\frac{\partial E_\omega}{\partial q^j}\dot{q}^j + \frac{\partial E_\omega}{\partial \dot{q}^j}\ddot{q}^j\right]. \tag{2.10}$$

E_ω is constant on TM if and only if $\mathrm{d}E_\omega(Y) = 0$ for all second-order vector fields Y on TM (see [7]).

If ω is conservative and Y is a Birkhoffian vector field, then (2.9) becomes

$$\mathrm{d}E_\omega(Y) = 0. \tag{2.11}$$

This means that E_ω is constant along the trajectories of Y.

We now introduce the concept of a dissipative Birkhoffian.

A *vertical 1-form on TM* (see for example [14]) is a 1-form Ψ on TM such that $\Psi(V^v) = 0$, for all V vector fields on M, where V^v is the vertical lift of the vector field V to TM. The local expression of a vertical 1-form is

$$\Psi = \sum_{j=1}^{m} \psi_j(q,\dot{q})\mathrm{d}q^j. \tag{2.12}$$

A 1-form D on TM is called *dissipative* if and only if D is vertical and $D(Y) > 0$, for all Y second-order vector fields on TM. Allowing for (2.12), the local expression of D is

$$D = \sum_{j=1}^{m} D_j(q,\dot{q})\mathrm{d}q^j \tag{2.13}$$

and from (2.13), (2.6), the inequality $D(Y) > 0$ becomes

$$\sum_{j=1}^{m} D_j(q,\dot{q})\dot{q}^j > 0. \tag{2.14}$$

A Birkhoffian ω of the configuration space M is called *dissipative* if and only if there exists a smooth function $E_{0_\omega} : TM \to \mathbf{R}$ such that

$$(X^*\omega)Y = \mathrm{d}E_{0_\omega}(Y) + D(Y) \tag{2.15}$$

for all second-order vector fields $Y = i \circ X$ on TM, D being a dissipative 1-form on TM.

Eq. (2.15) is equivalent, in a local coordinate system, to the identity

$$\sum_{j=1}^{m} Q_j(q\,\dot{q},\ddot{q})\dot{q}^j = \sum_{j=1}^{m}\left[\frac{\partial E_{0_\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j + D_j(q,\dot{q})\dot{q}^j\right]. \tag{2.16}$$

In view of (2.14), we obtain from (2.15),

$$(X^*\omega)Y > \mathrm{d}E_{0_\omega}(Y) \tag{2.17}$$

for all second-order vector fields $Y = i \circ X$. That is equivalent, in local coordinates, to the dissipation inequality

$$\sum_{j=1}^{m} Q_j(q\ \dot{q}, \ddot{q})\dot{q}^j > \sum_{j=1}^{m} \left[\frac{\partial E_{0\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j \right]. \tag{2.18}$$

If ω is a dissipative Birkhoffian and Y is the Birkhoffian vector field, then (2.17) becomes

$$\mathrm{d}E_{0_\omega}(Y) < 0. \tag{2.19}$$

This means that E_{0_ω} is nonincreasing along the trajectories of Y.

If the dissipative 1-form on TM has the particular expression

$$D = \sum_{i,j=1}^{m} \mathcal{D}_{ij}(q)\dot{q}^j \mathrm{d}q^i \tag{2.20}$$

when we calculate the function $D(Y)$ on TM, we obtain the so-called *Rayleigh dissipation function* $\mathcal{R} : TM \to \mathbf{R}$

$$\mathcal{R}(q, \dot{q}) = \sum_{i,j=1}^{m} \mathcal{D}_{ij}(q)\dot{q}^i\dot{q}^j. \tag{2.21}$$

3. RLC circuit dynamics

A simple electrical circuit provides us with an *oriented connected* graph, that is, a collection of points, called nodes, and a set of connecting lines or arcs, called branches, such that in each branch a direction is given and there is at least one path between any two nodes. A path is a sequence of branches such that the origin of the next branch coincides with the end of the previous one. The graph will be assumed to be *planar*, that is, it can be drawn in a plane without branches crossing. For the graph theoretic terminology, see for example [6].

Let b be the total number of branches in the graph, n be one less than the number of nodes and m be the cardinality of a selection of loops that cover the whole graph. Here, a loop is a path such that the first and last node coincide and that does not use the same branch more than once. By Euler's polyhedron formula, $b = m + n$.

We choose a reference node and a current direction in each l-branch of the graph, $l = 1, \ldots, b$. We also consider a covering of the graph with m loops, and a current direction in each j-loop, $j = 1, \ldots, m$. We assume that the associated graph has at least one loop, meaning that $m > 0$.

A graph can be described by matrices: a (bn)-matrix $B \in \mathfrak{M}_{bn}(\mathbf{R})$, $\operatorname{rank}(B) = n$, called the *incidence matrix* and a (bm)-matrix $A \in \mathfrak{M}_{bm}(\mathbf{R})$, $\operatorname{rank}(A) = m$, called the *loop matrix*. These matrices contain only 0, 1, −1. An element of the matrix B is 0 if a branch b is not incident with a node n, −1 if branch b enters node n and 1 if branch b leaves node n, respectively. An element of the matrix A is 0 if a branch b does not belong to a loop m, 1 if branch b belongs to loop m and their directions agree and −1 if branch b belongs to loop m and their directions oppose, respectively. For the fundamentals of electrical circuit theory, see for example [5].

The states of the circuit have two components, the currents through the branches, denoted by $\mathrm{I} \in \mathbf{R}^b$, and the voltages across the branches, denoted by $v \in \mathbf{R}^b$. Using the matrices A and B, Kirchhoff's current law and Kirchhoff's voltage law can be expressed by the equations

$$B^{\mathrm{T}}\mathrm{I} = 0 \quad \text{(KCL)} \tag{3.1}$$

$$A^{\mathrm{T}}v = 0 \quad \text{(KVL)}. \tag{3.2}$$

Tellegen's theorem establishes a relation between the matrices A^{T} and B^{T}: *the kernel of the matrix* B^{T} *is orthogonal to the kernel of the matrix* A^{T} (see, e.g., [4] page 5).

The next step is to introduce the branch elements in a simple electrical circuit. The branches of the graph associated with a RLC electrical circuit can be classified into three categories: resistive branches, inductor branches, and capacitor branches. Let r denote the number of resistive branches, k the number of inductor branches and p the number of capacitor branches, respectively. We assume that just one electrical device is associated with each branch, then, we have $b = r + k + p$. Thus, we can write $\mathrm{I} = (\mathrm{I}_{[\Gamma]}, \mathrm{I}_{(a)}, \mathrm{I}_{\alpha}) \in \mathbf{R}^r \times \mathbf{R}^k \times \mathbf{R}^p \simeq \mathbf{R}^b$, where $\mathrm{I}_{[\Gamma]}, \mathrm{I}_{(a)}, \mathrm{I}_{\alpha}$ are the currents through the resistors, the inductors, the capacitors, respectively, and $v = (v_{[\Gamma]}, v_{(a)}, v_{\alpha}) \in \mathbf{R}^r \times \mathbf{R}^k \times \mathbf{R}^p \simeq \mathbf{R}^b$, where $v_{[\Gamma]}, v_{(a)}, v_{\alpha}$ describe the voltage drops across the resistors, the inductors, the capacitors, respectively.
Each capacitor is supposed to be charge controlled. For the nonlinear capacitors we assume

$$v_{\alpha} = C_{\alpha}(\mathrm{Q}_{\alpha}), \quad \alpha = 1, \ldots, p \tag{3.3}$$

where the functions $C_{\alpha} : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ are smooth and invertible, and the Q_{α}'s denote the charges of the capacitors. The current through a capacitor is given by the time derivative of the corresponding charge

$$\mathrm{I}_{\alpha} = \frac{\mathrm{d}\mathrm{Q}_{\alpha}}{\mathrm{d}t}, \quad \alpha = 1, \ldots, p \tag{3.4}$$

where t is the time variable.
Each inductor is supposed to be current controlled. For the nonlinear inductors we assume

$$v_a = L_a(\mathrm{I}_a)\frac{\mathrm{d}\mathrm{I}_a}{\mathrm{d}t}, \quad a = 1, \ldots, k \tag{3.5}$$

where $L_a : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ are smooth invertible functions.
There are several types of nonlinear resistors, among them current controlled resistors and voltage controlled resistors. Generally, their constitutive relations are defined by some continuous functions of I_{Γ} and v_{Γ}, that is,
$f_{\Gamma}(\mathrm{I}_{\Gamma}, v_{\Gamma}) = 0, \quad \Gamma = 1, \ldots, r.$

3.1. Current controlled resistors

We first consider the case where the nonlinear resistors are current controlled, that is, the constitutive relations are given by

$$v_{\Gamma} = R_{\Gamma}(\mathrm{I}_{\Gamma}), \quad \Gamma = 1, \ldots, r \tag{3.6}$$

where $R_{\Gamma} : \mathbf{R} \longrightarrow \mathbf{R}$ are smooth functions. In order to obtain a dissipative Birkhoffian, we assume that, for all $x \neq 0$,

$$R_{\Gamma}(x)x > 0, \quad \forall \Gamma = 1, \ldots, r \tag{3.7}$$

that is, for each nonlinear resistor, the graph of the function R_{Γ} lies in the union of the first and the third quadrant.

For linear resistors the relations (3.6) can be written in the form

$$v_\Gamma = \mathrm{R}_\Gamma \mathrm{I}_\Gamma \tag{3.8}$$

where $\mathrm{R}_\Gamma > 0$ are real constants.
Taking into account (3.3)–(3.6), the Eqs. (3.1) and (3.2) become

$$\begin{cases} B^{\mathrm{T}} \begin{pmatrix} \mathrm{I}_\Gamma \\ \mathrm{I}_a \\ \dfrac{\mathrm{d}\mathrm{Q}_\alpha}{\mathrm{d}t} \end{pmatrix} = 0 \\ A^{\mathrm{T}} \begin{pmatrix} R_\Gamma(\mathrm{I}_\Gamma) \\ L_a(\mathrm{I}_a)\dfrac{\mathrm{d}\mathrm{I}_a}{\mathrm{d}t} \\ C_\alpha(\mathrm{Q}_\alpha) \end{pmatrix} = 0. \end{cases} \tag{3.9}$$

In the following we give a Birkhoffian formulation for the network described by the system of equations (3.9), *using the same procedure as in* [8]. *That is, using the first set of equations* (3.9), *we are going to define a family of m-dimensional affine–linear configuration spaces $M_c \subset \mathbf{R}^b$ parameterized by a constant vector c in $\mathbf{R}^n$. This vector is related to the initial values of the* Q-*variables at some instant of time. A Birkhoffian ω_c on the configuration space M_c arises from a linear combination of the second set of equations* (3.9). *Thus, (M_c, ω_c) will be a family of Birkhoff systems that describe the RLC network considered.*

We notice that the first set of equations (3.9) remains exactly the same for linear and nonlinear electrical devices. Thus, for obtaining the configuration space, it is not important whether the devices are linear or nonlinear.
Let $H : \mathbf{R}^b \longrightarrow \mathbf{R}^n$ be the linear map that, with respect to a coordinate system $(x^1, \ldots, x^b)$ on $\mathbf{R}^b$, is given by

$$H(x^1, \ldots, x^b) = B^{\mathrm{T}} \begin{pmatrix} x^1 \\ \vdots \\ x^b \end{pmatrix}. \tag{3.10}$$

Then, $H^{-1}(c)$, with c a constant vector in $\mathbf{R}^n$, is an affine–linear subspace in $\mathbf{R}^b$. Its dimension is $m = b - n$, because $\mathrm{rank}(B) = n$.
We define M_c as

$$M_c := H^{-1}(c). \tag{3.11}$$

We denote local coordinates on M_c by $q = (q^1, \ldots, q^m)$. Then, the natural coordinate system on the 2-jets bundle $J^2(M_c)$ is given by $(q, \dot{q}, \ddot{q})$.
We will now represent the Birkhoffian in a specific coordinate system on M_c:
In the vector spaces $\mathbf{R}^r$, $\mathbf{R}^k$, we identify points and vectors

$$\mathrm{I}_\Gamma := \frac{\mathrm{d}\mathrm{Q}_{[\Gamma]}}{\mathrm{d}t}, \quad \mathrm{I}_a := \frac{\mathrm{d}\mathrm{Q}_{(a)}}{\mathrm{d}t} \tag{3.12}$$

with $(\mathrm{Q}_{[\Gamma]})_{\Gamma=1,\ldots,r}$, $(\mathrm{Q}_{(a)})_{a=1,\ldots,k}$ coordinate systems on $\mathbf{R}^r$ and, respectively, on $\mathbf{R}^k$. Taking into account (3.12) and the fact that the matrix B^{T} is a constant matrix, we integrate the first set of equations (3.9) to arrive at

$$B^{\mathrm{T}} \begin{pmatrix} \mathrm{Q}_{[\Gamma]} \\ \mathrm{Q}_{(a)} \\ \mathrm{Q}_{\alpha} \end{pmatrix} = c \tag{3.13}$$

with c a constant vector in $\mathbf{R}^n$.
Likewise we consider coordinates in $\mathbf{R}^b \simeq \mathbf{R}^r \times \mathbf{R}^k \times \mathbf{R}^p$

$$\begin{aligned} &x^1 := \mathrm{Q}_{[1]}, \ldots, x^r := \mathrm{Q}_{[r]}, \quad x^{r+1} := \mathrm{Q}_{(1)}, \ldots, x^{r+k} := \mathrm{Q}_{(k)}, \\ &x^{r+k+1} := \mathrm{Q}_1, \ \ldots, x^b := \mathrm{Q}_p. \end{aligned} \tag{3.14}$$

From (3.10) and (3.11), we see that we can define coordinates on M_c by solving the equations (3.13) in terms of an appropriate set of m of the Q-variables, say $q = (q^1, \ldots, q^m)$. In other words, we express any of the x-variables as a function of $q = (q^1, \ldots, q^m)$, namely,

$$\begin{aligned} x^{\Gamma} &= \sum_{j=1}^{m} \mathcal{N}_j^{\Gamma} q^j + \text{const}, \quad \Gamma = 1, \ldots, r \\ x^{a} &= \sum_{j=1}^{m} \mathcal{N}_j^{a} q^j + \text{const}, \quad a = r+1, \ldots, r+k \\ x^{\alpha} &= \sum_{j=1}^{m} \mathcal{N}_j^{\alpha} q^j + \text{const}, \quad \alpha = r+k+1, \ldots, b \end{aligned} \tag{3.15}$$

with certain constants $\mathcal{N}_j^{\Gamma}, \mathcal{N}_j^{a}, \mathcal{N}_j^{\alpha}$. Here we can think of the constants const as being initial values of the x-variables at some instant of time.
From (3.4), (3.12), (3.14) and differentiating (3.15) we get

$$\mathrm{I} = \mathcal{N} \dot{q} \tag{3.16}$$

with the matrix of constants $\mathcal{N} \in \mathfrak{M}_{bm}(\mathbf{R})$, for some $\dot{q} \in \mathbf{R}^m$.
Using Tellegen's theorem and a fundamental theorem of linear algebra, we now find a relation between the matrices $\mathcal{N}$ and A.
By a fundamental theorem of linear algebra we have

$$(\mathrm{Ker}(A^{\mathrm{T}}))^{\perp} = \mathrm{Im}(A) \tag{3.17}$$

where $A \in \mathfrak{M}_{bm}(\mathbf{R})$, $\mathrm{Ker}(A^{\mathrm{T}}) := \{x \in \mathbf{R}^b \mid A^{\mathrm{T}} x = 0\}$ is the kernel of A^{T}, $\mathrm{Im}(A) := \{x \in \mathbf{R}^b \mid Ay = x$, for some $y \in \mathbf{R}^m\}$ is the image of A and $^{\perp}$ denotes the orthogonal complement in $\mathbf{R}^b$ of the respective vector subspace.
For the incidence matrix $B \in \mathfrak{M}_{bn}(\mathbf{R})$ and the loop matrix $A \in \mathfrak{M}_{bm}(\mathbf{R})$, which satisfy Kirchhoff's law (3.1) and (3.2), Tellegen's theorem can be written as

$$\mathrm{Ker}(B^{\mathrm{T}}) = (\mathrm{Ker}(A^{\mathrm{T}}))^{\perp}. \tag{3.18}$$

From the first set of equations in (3.9), and by construction of the matrix $\mathcal{N}$ in (3.16), we have

$$\mathrm{Ker}(B^{\mathrm{T}}) = \mathrm{Im}(\mathcal{N}). \tag{3.19}$$

Therefore, using (3.17)–(3.19), we obtain $\mathrm{Im}(A) = \mathrm{Im}(\mathcal{N})$. Then, another application of (3.17) yields

$$\mathrm{Ker}(A^{\mathrm{T}}) = \mathrm{Ker}(\mathcal{N}^{\mathrm{T}}). \tag{3.20}$$

Taking into account (3.20), we see that there exists a nonsingular matrix $\mathfrak{C} \in \mathfrak{M}_{mm}(\mathbf{R})$ satisfying

$$\mathfrak{C}A^{\mathrm{T}} = \mathcal{N}^{\mathrm{T}}. \tag{3.21}$$

The matrix $\mathfrak{C}$ provides a relation between the vector of the m independent loop currents and the coordinate vector q introduced on M_c.

Taking into account (3.20), *we define the Birkhoffian* ω_c *of* M_c *such that the differential system* (2.5) *is the linear combination of the second set of equations in* (3.9) *obtained by replacing* A^{T} *with the matrix* $\mathcal{N}^{\mathrm{T}}$. *Thus, in terms of q-coordinates as chosen before, the expressions of the components* $Q_j(q, \dot{q}, \ddot{q})$ *are*

$$Q_j(q, \dot{q}, \ddot{q}) = F_j(\dot{q})\ddot{q} + H_j(\dot{q}) + G_j(q), \quad j = 1, \ldots, m \tag{3.22}$$

where

$$\begin{aligned} F_j(\dot{q})\ddot{q} &:= \sum_{a=r+1}^{r+k} \mathcal{N}_j^a L_{a-r}\left(\sum_{l=1}^{m} \mathcal{N}_l^a \dot{q}^l\right)\left(\sum_{i=1}^{m} \mathcal{N}_i^a \ddot{q}^i\right) \\ &= \sum_{i=1}^{m} \sum_{a=r+1}^{r+k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_{a-r}(\dot{q})\, \ddot{q}^i \end{aligned} \tag{3.23}$$

$$H_j(\dot{q}) := \sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} R_{\Gamma}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\Gamma} \dot{q}^l\right) = \sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} \widetilde{R}_{\Gamma}(\dot{q}) \tag{3.24}$$

$$G_j(q) := \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^{\alpha} C_{\alpha-r-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\alpha} q^l + \mathrm{const}\right) = \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^{\alpha} \widetilde{C}_{\alpha-r-k}(q). \tag{3.25}$$

We note that the Birkhoffian (3.22) is *not* conservative. We easily see that there does not exist a function E_ω such that (2.10) is fulfilled for the Birkhoffian (3.22), since $\frac{\partial^2 E_\omega}{\partial q^j \partial \dot{q}^j} \neq \frac{\partial^2 E_\omega}{\partial \dot{q}^j \partial q^j}$.

For an RLC electrical network with nonlinear resistors, described by (3.6) *and* (3.7), *nonlinear inductors and capacitors described by* (3.5), *respectively* (3.3), *we claim that the Birkhoffian* (3.22) *is a* dissipative Birkhoffian.

Indeed, in the view of the assumption (3.7), the vertical 1-form D on TM given by

$$D = \sum_{j=1}^{m} H_j(\dot{q}) \mathrm{d}q^j \tag{3.26}$$

with $H_j(\dot{q})$ in (3.24), is dissipative, that is,

$$\sum_{j=1}^{m} H_j(\dot{q})\dot{q}^j = \sum_{\Gamma=1}^{r}\left[R_{\Gamma}\left(\sum_{j=1}^{m} \mathcal{N}_j^{\Gamma} \dot{q}^j\right)\right]\left(\sum_{j=1}^{m} \mathcal{N}_j^{\Gamma} \dot{q}^j\right) > 0. \tag{3.27}$$

We showed in [8] that the following smooth function E_{0_ω} on TM

$$E_{0_\omega} = \sum_{a=r+1}^{r+k} \sum_{l=1}^{m} \sum_{i_1<\cdots<i_l=1}^{m} (-1)^{l+1} \underbrace{\int\cdots\int}_{l} \Big[\widetilde{L}_{a-r}^{(l-1)}(\dot{q})\mathcal{N}_i^a \dot{q}^i + (l-1)\widetilde{L}_a^{(l-2)}(\dot{q})\Big]\mathcal{N}_{i_1}^a \cdots \mathcal{N}_{i_l}^a \mathrm{d}\dot{q}^{i_1}\cdots \mathrm{d}\dot{q}^{i_l} + \sum_{\alpha=r+k+1}^{b} \sum_{l=1}^{m} \sum_{i_1<\cdots<i_l=1}^{m} (-1)^{l+1} \underbrace{\int\cdots\int}_{l} \widetilde{C}_{\alpha-r-k}^{(l-1)}(q)\mathcal{N}_{i_1}^\alpha \cdots \mathcal{N}_{i_l}^\alpha \mathrm{d}q^{i_1}\cdots \mathrm{d}q^{i_l} \quad (3.28)$$

satisfies the identity

$$\sum_{j=1}^{m} \big[F_j(\dot{q})\ddot{q} + G_j(q)\big]\dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial E_{0_\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j\right]. \quad (3.29)$$

According to (3.29), the Birkhoffian (3.22) satisfies (2.16) with the function $E_{0_\omega}(q, \dot{q})$ given by (3.28) and the dissipative 1-form D given by (3.26). □

Let us now discuss the regularity of the Birkhoffian given by (3.22).

If there exists in the network at least one loop that contains only capacitors, or only resistors, or only resistors and capacitors, then the Birkhoffian (3.22) *associated with the network is* never regular.

In [8] we have shown that if there exists at least one loop in an LC network that contains only capacitors, then the Birkhoffian associated with the network is never regular. The Birkhoffian associated with an RLC network which contains at least one loop formed only by resistors or only by resistors and capacitors is never regular as well. The proof is based on the fact that for the l-loop which does not contain any inductor branches, for the column l of the matrix A we have

$$A_l^a = 0, \quad a = r+1, \ldots, r+k. \quad (3.30)$$

For the Birkhoffian (3.22), the determinant in (2.8) becomes

$$\det\left[\frac{\partial Q_j}{\partial \ddot{q}^i}(q, \dot{q}, \ddot{q})\right]_{i,j=1,\ldots,m} = \det\left[\sum_{a=r+1}^{r+k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_{a-r}(\dot{q})\right]_{i,j=1,\ldots,m}. \quad (3.31)$$

From (3.21), we get $\mathcal{N}_j^a = \sum_{i_1=1}^{m} \mathfrak{C}_j^{i_1} A_{i_1}^a$, for any $a = r+1, \ldots, r+k$ and taking into account (3.30), we have

$$\sum_{a=r+1}^{r+k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_{a-r}(\dot{q}) = \sum_{\substack{i_1=1\\ i_1\neq l}}^{m} \mathfrak{C}_j^{i_1}\mathfrak{C}_i^{i_1}\left[\sum_{a=r+1}^{r+k} (A_{i_1}^a)^2 \widetilde{L}_{a-r}(\dot{q})\right] + \sum_{\substack{i_1<j_1\\ i_1,j_1\neq l}}^{m} \left(\mathfrak{C}_j^{i_1}\mathfrak{C}_i^{j_1} + \mathfrak{C}_i^{i_1}\mathfrak{C}_j^{j_1}\right)\left[\sum_{a=r+1}^{r+k} A_{i_1}^a A_{j_1}^a \widetilde{L}_{a-r}(\dot{q})\right]. \quad (3.32)$$

Using basic calculus, the determinant of the matrix with the elements (3.32) is a linear combination of determinants having at least two linearly dependent columns. This shows that the determinant in the right-hand side of (3.31) is equal to zero. Thus, the Birkhoffian (3.22) is not regular. □

We now discuss the question of how to proceed in the case where the Birkhoffian given by (3.22) is not regular in the sense of definition (2.8).

If there exist in the network $m_1 < m$ loops which contain only capacitors, all the other loops containing at least an inductor, we can regularize *the Birkhoffian* (3.22) *via* reduction of the configuration space. *The reduced configuration space $\bar{M}_c$, of dimension $m - m_1$, is a linear or a nonlinear subspace of M_c, depending on whether the capacitors are linear or nonlinear. We claim that the Birkhoffian $\bar{\omega}_c$ on the reduced configuration space $\bar{M}_c$ is still a* dissipative Birkhoffian. *Under certain conditions on the functions L_a, $a = 1, \dots, k$, which characterize the inductors, the reduced Birkhoffian $\bar{\omega}_c$ will be a* regular Birkhoffian.

Without loss of generality, we can assume that there is one loop in the network that contains only capacitors and in the coordinate system we have chosen

$$\mathcal{N}_1^{\Gamma} = 0, \quad \Gamma = 1, \dots, r, \quad \mathcal{N}_1^{a} = 0, \quad a = r+1, \dots, r+k. \tag{3.33}$$

Thus, the Birkhoffian components (3.22), with (3.23)–(3.25), are given by, $j = 2, \dots, m$,

$$\begin{aligned} Q_1(q, \dot{q}, \ddot{q}) &= \sum_{\alpha=r+k+1}^{b} \mathcal{N}_1^{\alpha} \widetilde{C}_{\alpha-r-k}(q) \\ Q_j(q, \dot{q}, \ddot{q}) &= \sum_{i=2}^{m} \sum_{a=r+1}^{r+k} \mathcal{N}_j^{a} \mathcal{N}_i^{a} \widetilde{L}_{a-r}(\dot{q})\, \ddot{q}^i + \sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} \widetilde{R}_{\Gamma}(\dot{q}) \\ &\quad + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^{\alpha} \widetilde{C}_{\alpha-r-k}(q). \end{aligned} \tag{3.34}$$

We note that, according to (3.33), $\dot{q}^1$ does not appear in any function $\widetilde{R}_{\Gamma}(\dot{q})$, $\widetilde{L}_{a-r}(\dot{q})$ and the terms $\widetilde{L}_{a-r}(\dot{q})\ddot{q}^1$ do not appear in any of the Birkhoffian components $Q_2(q, \dot{q}, \ddot{q}), \dots, Q_m(q, \dot{q}, \ddot{q})$.

We define the $(m-1)$-dimensional nonlinear space $\bar{M}_c \subset M_c$ by

$$\bar{M}_c = \left\{ q \in M_c \,\middle|\, \sum_{\alpha=r+k+1}^{b} \mathcal{N}_1^{\alpha} \widetilde{C}_{\alpha-r-k}(q) = 0 \right\}. \tag{3.35}$$

By the implicit function theorem, we obtain a local coordinate system on the reduced configuration space $\bar{M}_c$. Taking $\bar{q}^1 := q^2, \dots, \bar{q}^{m-1} := q^m$, the Birkhoffian has the form $\bar{\omega}_c = \sum_{j=1}^{m-1} \bar{Q}_j \mathrm{d}\bar{q}^j$,

$$\bar{Q}_j(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) = \bar{F}_j(\dot{\bar{q}})\ddot{\bar{q}} + \bar{H}_j(\dot{\bar{q}}) + \bar{G}_j(\bar{q}), \quad \text{where} \tag{3.36}$$

$$\bar{F}_j(\dot{\bar{q}})\ddot{\bar{q}} := \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}_{(j+1)}^{a} \mathcal{N}_{(i+1)}^{a} L_{a-r} \left(\sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^{a} \dot{\bar{q}}^l \right) \ddot{\bar{q}}^i \tag{3.37}$$

$$\bar{H}_j(\dot{\bar{q}}) := \sum_{\Gamma=1}^{r} \mathcal{N}_{(j+1)}^{\Gamma} R_{\Gamma} \left(\sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^{\Gamma} \dot{\bar{q}}^l \right) \tag{3.38}$$

$$\bar{G}_j(\bar{q}) := \sum_{\alpha=r+k+1}^{b} \mathcal{N}^{\alpha}_{(j+1)} C_{\alpha-r-k} \left(\mathcal{N}^{\alpha}_{1} f(\bar{q}^1, \ldots, \bar{q}^{m-1}) + \sum_{l=1}^{m-1} \mathcal{N}^{\alpha}_{(l+1)} \bar{q}^l + \text{const} \right) \tag{3.39}$$

where $f : U \subset \mathbf{R}^{m-1} \longrightarrow \mathbf{R}$ is the unique function such that $f(\bar{q}_0) = q_0^1$, $q_0^1 \in \mathbf{R}$, and

$$\sum_{\alpha=r+k+1}^{b} \mathcal{N}^{\alpha}_{1} C_{\alpha-r-k} \left(\mathcal{N}^{\alpha}_{1} f(\bar{q}^1, \ldots, \bar{q}^{m-1}) + \sum_{l=1}^{m-1} \mathcal{N}^{\alpha}_{(l+1)} \bar{q}^l + \text{const} \right) = 0 \tag{3.40}$$

for all $\bar{q} = (\bar{q}^1, \ldots, \bar{q}^{m-1}) \in U$, with U a neighborhood of $\bar{q}_0 = (\bar{q}_0^1, \ldots, \bar{q}_0^{m-1})$.

We will now prove that the Birkhoffian (3.36) is dissipative. In order to do so, we will show that there exists a function $\bar{E}_{0_\omega}(\bar{q}, \dot{\bar{q}})$ satisfying

$$\sum_{j=1}^{m-1} \bar{Q}_j(\bar{q}\dot{\bar{q}}, \ddot{\bar{q}}) \dot{\bar{q}}^j = \sum_{j=1}^{m-1} \left[\frac{\partial \bar{E}_{0_\omega}}{\partial \bar{q}^j} \dot{\bar{q}}^j + \frac{\partial \bar{E}_{0_\omega}}{\partial \dot{\bar{q}}^j} \ddot{\bar{q}}^j + \bar{D}_j(\bar{q}, \dot{\bar{q}}) \dot{\bar{q}}^j \right] \tag{3.41}$$

where $\bar{D} = \sum_{j=1}^{m-1} \bar{D}_j(\bar{q}, \dot{\bar{q}}) \mathrm{d}\bar{q}^j$ is a dissipative 1-form on $T\bar{M}_c$.

We consider the following 1-form on $T\bar{M}_c$

$$\bar{D} = \sum_{j=1}^{m-1} \sum_{\Gamma=1}^{r} \bar{H}_j(\dot{\bar{q}}) \mathrm{d}\bar{q}^j. \tag{3.42}$$

In the view of the assumption (3.7), the vertical 1-form (3.42) on $T\bar{M}_c$ is dissipative, that is,

$$\sum_{\Gamma=1}^{r} \left[R_\Gamma \left(\sum_{j=1}^{m-1} \mathcal{N}^{\Gamma}_{(j+1)} \dot{\bar{q}}^j \right) \right] \left(\sum_{j=1}^{m-1} \mathcal{N}^{\Gamma}_{(j+1)} \dot{\bar{q}}^j \right) > 0. \tag{3.43}$$

Therefore, (3.41) is fulfilled if $\bar{E}_{0_\omega}(\bar{q}, \dot{\bar{q}})$ can be chosen in such a way that

$$\sum_{j=1}^{m-1} \left[\bar{F}_j(\dot{\bar{q}}) \ddot{\bar{q}} + \bar{G}_j(\bar{q}) \right] \dot{\bar{q}}^j = \sum_{j=1}^{m-1} \left[\frac{\partial \bar{E}_{0_\omega}}{\partial \bar{q}^j} \dot{\bar{q}}^j + \frac{\partial \bar{E}_{0_\omega}}{\partial \dot{\bar{q}}^j} \ddot{\bar{q}}^j \right] \tag{3.44}$$

is satisfied. Because of the special form of the terms on the left side of (3.44), we may assume that $\bar{E}_{0_\omega}(\bar{q}, \dot{\bar{q}})$ is a sum of a function depending only on $\bar{q}$, and a function depending only on $\dot{\bar{q}}$. From the theory of total differentials, a necessary condition for the existence of such functions is the fulfilment of the following relations

$$\begin{cases} \dfrac{\partial \bar{\mathcal{F}}_j(\dot{\bar{q}})}{\partial \dot{\bar{q}}^l} - \dfrac{\partial \bar{\mathcal{F}}_l(\dot{\bar{q}})}{\partial \dot{\bar{q}}^j} = 0 \\ \dfrac{\partial \bar{G}_j(\bar{q})}{\partial \bar{q}^l} - \dfrac{\partial \bar{G}_l(\bar{q})}{\partial \bar{q}^j} = 0 \end{cases} \tag{3.45}$$

for any $j, l = 1, \ldots, m-1$, where

$$\bar{\mathcal{F}}_j(\dot{\bar{q}}) := \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}^{a}_{(j+1)} \mathcal{N}^{a}_{(i+1)} L_{a-r} \left(\sum_{l=1}^{m-1} \mathcal{N}^{a}_{(l+1)} \dot{\bar{q}}^l \right) \dot{\bar{q}}^i. \tag{3.46}$$

From (3.46), we get:

$$\frac{\partial \bar{\mathcal{F}}_j(\dot{\bar{q}})}{\partial \dot{\bar{q}}^l} = \sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(l+1)}\widetilde{L}_{a-r}(\dot{\bar{q}}) + \sum_{i=1}^{m-1}\sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(i+1)}\mathcal{N}^a_{(l+1)}\widetilde{L}'_{a-r}(\dot{\bar{q}})\dot{\bar{q}}^i \tag{3.47}$$

where $\widetilde{L}'_{a-r} := \frac{\mathrm{d}\widetilde{L}_{a-r}(\eta)}{\mathrm{d}\eta}$. Then, the left side of the first relation in (3.45) becomes

$$\sum_{a=r+1}^{r+k}\left[\left(\mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(l+1)} - \mathcal{N}^a_{(l+1)}\mathcal{N}^a_{(j+1)}\right)\widetilde{L}_{a-r}(\dot{\hat{q}}) + \left(\mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(l+1)} - \mathcal{N}^a_{(l+1)}\mathcal{N}^a_{(j+1)}\right)\left(\sum_{i=1}^{m-1}\mathcal{N}^a_{(i+1)}\dot{\hat{q}}^i\right)\widetilde{L}'_{a-r}(\dot{\hat{q}})\right]. \tag{3.48}$$

We easily see that the expression in (3.48) is zero, thus the first relation in (3.45) is fulfilled. From (3.39), the second relation in (3.45) reads

$$\sum_{\alpha=r+k+1}^{b}\left\{\mathcal{N}^\alpha_{(j+1)}\widetilde{C}'_{\alpha-r-k}(\bar{q})\left[\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^l} + \mathcal{N}^\alpha_{(l+1)}\right] - \mathcal{N}^\alpha_{(l+1)}\widetilde{C}'_{\alpha-r-k}(\bar{q})\left[\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^j} + \mathcal{N}^\alpha_{(j+1)}\right]\right\} = 0 \tag{3.49}$$

where $\widetilde{C}'_{\alpha-r-k} := \frac{\mathrm{d}\widetilde{C}_{\alpha-r-k}(\eta)}{\mathrm{d}\eta}$. The relation (3.49) reduces to

$$\sum_{\alpha=r+k+1}^{b}\mathcal{N}^\alpha_{(j+1)}\widetilde{C}'_{\alpha-r-k}(\bar{q})\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^l} - \mathcal{N}^\alpha_{(l+1)}\widetilde{C}'_{\alpha-r-k}(\bar{q})\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^j} = 0. \tag{3.50}$$

Taking into account (3.40), the above relation is fulfilled, for any $j, l = 1, \ldots, m-1$. Indeed, taking the derivatives with respect to $\bar{q}^j$ and also $\bar{q}^l$, in Eq. (3.40), we obtain, respectively,

$$\begin{aligned}&\sum_{\alpha=r+k+1}^{b}\mathcal{N}^\alpha_1\widetilde{C}'_{\alpha-r-k}(\bar{q})\left[\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^j} + \mathcal{N}^\alpha_{(j+1)}\right] = 0\\ &\sum_{\alpha=r+k+1}^{b}\mathcal{N}^\alpha_1\widetilde{C}'_{\alpha-r-k}(\bar{q})\left[\mathcal{N}^\alpha_1\frac{\partial f(\bar{q})}{\partial \bar{q}^l} + \mathcal{N}^\alpha_{(l+1)}\right] = 0.\end{aligned} \tag{3.51}$$

Now we multiply in (3.51) the first equation with $\frac{\partial f(\bar{q})}{\partial \bar{q}^l}$, the second equation with $-\frac{\partial f(\bar{q})}{\partial \bar{q}^j}$ and we add the resulting equations to obtain the Eq. (3.50).

Thus, we have proved the existence of a function $\bar{E}_{0_\omega}(\bar{q}, \dot{\bar{q}})$ such that (3.41) is fulfilled, with the dissipative 1-form given by (3.42). Therefore, the Birkhoffian (3.36) is dissipative.

For the Birkhoffian (3.36), the determinant in (2.8) becomes

$$\det\left[\frac{\partial \bar{Q}_j}{\partial \ddot{\bar{q}}^i}(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}})\right]_{i,j=1,\ldots,m-1} = \det\left[\sum_{a=r+1}^{r+k}\mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(i+1)}\widetilde{L}_{a-r}\left(\dot{\bar{q}}\right)\right]. \tag{3.52}$$

If the determinant in (3.52) is different from zero, then the Birkhoffian (3.36) is regular. □

If there exist in the network $m_2 < m$ loops which contain only resistors, all the other loops containing at least an inductor, we can regularize *the Birkhoffian* (3.22) *via* reduction of the configuration space. *The reduced configuration space $\hat{M}_c$ of dimension $m - m_2$ is a linear or a nonlinear subspace of M_c, depending on whether the resistors are linear or nonlinear. We claim that the Birkhoffian $\hat{\omega}_c$ on the reduced configuration space $\hat{M}_c$ is still a* dissipative Birkhoffian. *Under certain conditions on the functions L_a, $a = 1, \ldots, k$, which characterize the inductors, the reduced Birkhoffian $\hat{\omega}_c$ will be a* regular Birkhoffian.

Without loss of generality, we may assume that we have one loop in the network that contains only resistors and in the coordinate system that we have chosen, the constants read as

$$\mathcal{N}_1^a = 0, \quad a = r+1, \ldots, r+k, \qquad \mathcal{N}_1^\alpha = 0, \ \alpha = r+k+1, \ldots, b. \tag{3.53}$$

(I) Let us first consider the case where the resistors in this loop are linear resistors, that is, described by (3.8), all the other electrical devices in the network being nonlinear. This means that we have

$$\mathcal{N}_1^\Gamma \neq 0, \quad \Gamma = 1, \ldots, r_{\text{lin}}, \qquad \mathcal{N}_1^\Gamma = 0, \ \Gamma = r_{\text{lin}}+1, \ldots, r \tag{3.54}$$

where r_{lin} is the number of linear resistors in the network.
In this case, the expressions of the Birkhoffian components (3.22), with (3.23)–(3.25), are given by, $j = 2, \ldots, m$,

$$\begin{aligned} Q_1(q, \dot{q}, \ddot{q}) &= \sum_{\Gamma=1}^{r_{\text{lin}}} \sum_{l=1}^{m} \mathcal{N}_1^\Gamma \mathrm{R}_\Gamma \mathcal{N}_l^\Gamma \dot{q}^l \\ Q_j(q, \dot{q}, \ddot{q}) &= \sum_{i=2}^{m} \sum_{a=r+1}^{r+k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_{a-r}(\dot{q})\, \ddot{q}^i + \sum_{\Gamma=1}^{r_{\text{lin}}} \sum_{l=1}^{m} \mathcal{N}_j^\Gamma \mathrm{R}_\Gamma \mathcal{N}_l^\Gamma \dot{q}^l \\ &\quad + \sum_{\Gamma=r_{\text{lin}}+1}^{r} \mathcal{N}_j^\Gamma \widetilde{R}_\Gamma(\dot{q}) + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^\alpha \widetilde{C}_{\alpha-r-k}(q). \end{aligned} \tag{3.55}$$

We note that according to (3.53) and (3.54), $\dot{q}^1$ does not appear in any function $\widetilde{R}_\Gamma(\dot{q})$, $\widetilde{L}_{a-r}(\dot{q})$, the terms $\widetilde{L}_{a-r}(\dot{q})\ddot{q}^1$ do not appear in any of the Birkhoffian components $Q_2(q, \dot{q}, \ddot{q}), \ldots, Q_m(q, \dot{q}, \ddot{q})$ and q^1 does not appear in any function $\widetilde{C}_{\alpha-r-k}(q)$.
We define the $(m-1)$-dimensional linear space $\hat{M}_c \subset M_c$ by

$$\hat{M}_c = \left\{ q \in M_c \,\middle|\, \sum_{\Gamma=1}^{r_{\text{lin}}} \sum_{l=1}^{m} \mathcal{N}_1^\Gamma \mathrm{R}_\Gamma \mathcal{N}_l^\Gamma q^l + c_1 = 0 \right\} \tag{3.56}$$

with c_1 a real constant.
We take $\hat{q}^1 := q^2, \ldots, \hat{q}^{m-1} := q^m$ as local coordinates on the reduced configuration space $\hat{M}_c$. Then, making use of (3.56) and of the fact that $\mathcal{N}_1^\Gamma \neq 0$ and $\mathrm{R}_\Gamma > 0$, for any $\Gamma = 1, \ldots, r_{\text{lin}}$, we can express $\dot{q}^1$ as a linear combination of $\dot{\hat{q}}^1, \ldots, \dot{\hat{q}}^{m-1}$, denoted as $g(\dot{\hat{q}})$, such that

$$\sum_{\Gamma=1}^{r_{\text{lin}}} \mathcal{N}_1^\Gamma \mathrm{R}_\Gamma \left[\mathcal{N}_1^\Gamma g(\dot{\hat{q}}) + \sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^\Gamma \dot{\hat{q}}^l \right] = 0. \tag{3.57}$$

Thus, the reduced Birkhoffian has the form $\hat{\omega}_c = \sum_{j=1}^{m-1} \hat{Q}_j \mathrm{d}\hat{q}^j$,

$$\hat{Q}_j(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}}) = \hat{F}_j(\dot{\hat{q}})\ddot{\hat{q}} + \hat{H}_j(\dot{\hat{q}}) + \hat{G}_j(\hat{q}), \quad \text{where} \tag{3.58}$$

$$\hat{F}_j(\dot{\hat{q}})\ddot{\hat{q}} := \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)} \mathcal{N}^a_{(i+1)} L_{a-r} \left(\sum_{l=1}^{m-1} \mathcal{N}^a_{(l+1)} \dot{\hat{q}}^l \right) \ddot{\hat{q}}^i \tag{3.59}$$

$$\hat{H}_j(\dot{\hat{q}}) := \sum_{\Gamma=1}^{r_{\text{lin}}} \mathcal{N}^\Gamma_{(j+1)} \mathrm{R}_\Gamma \left[\mathcal{N}^\Gamma_1 g(\dot{\hat{q}}) + \sum_{l=1}^{m-1} \mathcal{N}^\Gamma_{(l+1)} \dot{\hat{q}}^l \right] + \sum_{\Gamma=r_{\text{lin}}+1}^{r} \mathcal{N}^\Gamma_{(j+1)} R_\Gamma \left(\sum_{l=1}^{m-1} \mathcal{N}^\Gamma_{(l+1)} \dot{\hat{q}}^l \right) \tag{3.60}$$

$$\hat{G}_j(\hat{q}) := \sum_{\alpha=r+k+1}^{b} \mathcal{N}^\alpha_{(j+1)} C_{\alpha-r-k} \left(\sum_{l=1}^{m-1} \mathcal{N}^\alpha_{(l+1)} \hat{q}^l + \text{const} \right). \tag{3.61}$$

The Birkhoffian given by (3.58) is still dissipative. We will see that there exists a function $\hat{E}_{0_\omega}(\hat{q}, \dot{\hat{q}})$ such that

$$\sum_{j=1}^{m-1} \hat{Q}_j(\hat{q}\, \dot{\hat{q}}, \ddot{\hat{q}})\dot{\hat{q}}^j = \sum_{j=1}^{m-1} \left[\frac{\partial \hat{E}_{0_\omega}}{\partial \hat{q}^j} \dot{\hat{q}}^j + \frac{\partial \hat{E}_{0_\omega}}{\partial \dot{\hat{q}}^j} \ddot{\hat{q}}^j + \hat{D}_j(\hat{q}, \dot{\hat{q}})\dot{\hat{q}}^j \right] \tag{3.62}$$

where $\hat{D} = \sum_{j=1}^{m-1} \hat{D}_j(\hat{q}, \dot{\hat{q}})\mathrm{d}\hat{q}^j$ is a dissipative 1-form on $T\hat{M}_c$.
We consider the following 1-form on $T\hat{M}_c$

$$\hat{D} = \sum_{j=1}^{m-1} \hat{H}_j(\dot{\hat{q}})\mathrm{d}\hat{q}^j. \tag{3.63}$$

Let us check that the vertical 1-form (3.63) is dissipative, that is,

$$\sum_{j=1}^{m-1} \hat{H}_j(\dot{\hat{q}})\dot{\hat{q}}^j > 0. \tag{3.64}$$

From (3.60), the left side of (3.64) can be written as the sum $\mathcal{S}_1 + \mathcal{S}_2$, where

$$\mathcal{S}_1 = \sum_{j=1}^{m-1} \sum_{\Gamma=1}^{r_{\text{lin}}} \mathcal{N}^\Gamma_{(j+1)}\, \mathrm{R}_\Gamma \left[\mathcal{N}^\Gamma_1 g(\dot{\hat{q}}) + \sum_{l=1}^{m-1} \mathcal{N}^\Gamma_{(l+1)} \dot{\hat{q}}^l \right] \dot{\hat{q}}^j \tag{3.65}$$

$$\mathcal{S}_2 = \sum_{\Gamma=r_{\text{lin}}+1}^{r} \left[R_\Gamma \left(\sum_{j=1}^{m-1} \mathcal{N}^\Gamma_{(j+1)} \dot{\hat{q}}^j \right) \right] \left(\sum_{j=1}^{m-1} \mathcal{N}^\Gamma_{(j+1)} \dot{\hat{q}}^j \right). \tag{3.66}$$

We now multiply the Eq. (3.57) by the function $g(\dot{\hat{q}})$. Using the resulting equation we can write the sum in (3.65) in the form

$$\mathcal{S}_1 = \sum_{\Gamma=1}^{r_{\text{lin}}} \mathrm{R}_\Gamma \left[\mathcal{N}^\Gamma_1 g(\dot{\hat{q}}) + \sum_{l=1}^{m-1} \mathcal{N}^\Gamma_{(l+1)} \dot{\hat{q}}^l \right]^2. \tag{3.67}$$

Since $\mathrm{R}_\Gamma > 0$, $\Gamma = 1, \dots, r_{\text{lin}}$, the sum $\mathcal{S}_1$ is strictly positive.
Because all nonlinear resistors considered satisfy the condition (3.7), the sum $\mathcal{S}_2$ in (3.66) is strictly positive as well. Therefore, the inequality (3.64) is fulfilled and the vertical 1-form in (3.63) is dissipative.
We now look for a function $\hat{E}_{0_\omega}(\hat{q}, \dot{\hat{q}})$ such that

$$\sum_{j=1}^{m-1} \left[\hat{F}_j(\dot{\hat{q}})\ddot{\hat{q}} + \hat{G}_j(\hat{q}) \right] \dot{\hat{q}}^j = \sum_{j=1}^{m-1} \left[\frac{\partial \hat{E}_{0_\omega}}{\partial \hat{q}^j} \dot{\hat{q}}^j + \frac{\partial \hat{E}_{0_\omega}}{\partial \dot{\hat{q}}^j} \ddot{\hat{q}}^j \right]. \tag{3.68}$$

Because of the special form of the terms on the left side of (3.68), we can look for the required function $\hat{E}_{0_\omega}(\hat{q}, \dot{\hat{q}})$ as a sum of a function only depending on $\hat{q}$, and a function only depending on $\dot{\hat{q}}$. From the theory of total differentials, a necessary condition for the existence of such functions is the fulfilment of the following relations

$$\begin{cases} \dfrac{\partial \hat{\mathcal{F}}_j(\dot{\hat{q}})}{\partial \dot{\hat{q}}^l} - \dfrac{\partial \hat{\mathcal{F}}_l(\dot{\hat{q}})}{\partial \dot{\hat{q}}^j} = 0 \\ \dfrac{\partial \hat{G}_j(\hat{q})}{\partial \hat{q}^l} - \dfrac{\partial \hat{G}_l(\hat{q})}{\partial \hat{q}^j} = 0 \end{cases} \tag{3.69}$$

for any $j, l = 1, \dots, m-1$, where

$$\hat{\mathcal{F}}_j(\dot{\hat{q}}) = \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)} \mathcal{N}^a_{(i+1)} L_{a-r} \left(\sum_{l=1}^{m-1} \mathcal{N}^a_{(l+1)} \dot{\hat{q}}^l \right) \dot{\hat{q}}^i. \tag{3.70}$$

From (3.70) and (3.61) we get

$$\frac{\partial \hat{\mathcal{F}}_j(\dot{\hat{q}})}{\partial \dot{\hat{q}}^l} = \sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)} \mathcal{N}^a_{(l+1)} \widetilde{L}_{a-r}(\dot{\hat{q}}) + \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)} \mathcal{N}^a_{(i+1)} \mathcal{N}^a_{(l+1)} \widetilde{L}'_{a-r}(\dot{\hat{q}}) \dot{\hat{q}}^i \tag{3.71}$$

$$\frac{\partial \hat{G}_j(\hat{q})}{\partial \hat{q}^l} = \sum_{\alpha=r+k+1}^{b} \mathcal{N}^\alpha_{(j+1)} \mathcal{N}^\alpha_{(l+1)} \widetilde{C}'_{\alpha-r-k}(\hat{q}) \tag{3.72}$$

where $\widetilde{L}'_{a-r} := \frac{\mathrm{d}\widetilde{L}_{a-r}(\eta)}{\mathrm{d}\eta}$, $\widetilde{C}'_{\alpha-r-k} := \frac{\mathrm{d}\widetilde{C}_{\alpha-r-k}(\eta)}{\mathrm{d}\eta}$. We can easily check that Eqs. (3.69) are fulfilled. Thus, we proved the existence of a function $\hat{E}_{0_\omega}(\hat{q}, \dot{\hat{q}})$ such that (3.62) is fulfilled, with the dissipative 1-form given by (3.63), that is, the Birkhoffian (3.58) is dissipative.

For the Birkhoffian (3.58), the determinant in (2.8) becomes

$$\det \left[\frac{\partial \hat{Q}_j}{\partial \ddot{\hat{q}}^i}(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}}) \right]_{i,j=1,\dots,m-1} = \det \left[\sum_{a=r+1}^{r+k} \mathcal{N}^a_{(j+1)} \mathcal{N}^a_{(i+1)} \widetilde{L}_{a-r} \left(\dot{\hat{q}} \right) \right]. \tag{3.73}$$

If the determinant in (3.73) is different from zero, then the Birkhoffian (3.58) is regular. □

(II) Let us now consider the case where the resistors in the loop formed only by resistors are nonlinear devices too, that is, they are described by (3.6), with the assumption (3.7). Now, instead of (3.54) we have $\mathcal{N}_1^\Gamma \neq 0$, $\Gamma = 1, \dots, r$.

The component $Q_1(q, \dot{q}, \ddot{q})$ of the Birkhoffian takes the form

$$Q_1(q, \dot{q}, \ddot{q}) = \sum_{\Gamma=1}^{r} \mathcal{N}_1^{\Gamma} \widetilde{R}_{\Gamma}(\dot{q}) \tag{3.74}$$

and the other components $Q_2(q, \dot{q}, \ddot{q}), \ldots, Q_m(q, \dot{q}, \ddot{q})$ are the same as in (3.55) with the terms following $r_{\text{lin}} = 0$ absent. According to (3.53), $\dot{q}^1$ does not appear in any function $\widetilde{L}_{a-r}(\dot{q})$, the terms $\widetilde{L}_{a-r}(\dot{q})\ddot{q}^1$ do not appear in any of the Birkhoffian components $Q_2(q, \dot{q}, \ddot{q}), \ldots, Q_m(q, \dot{q}, \ddot{q})$ and q^1 does not appear in any function $\widetilde{C}_{\alpha-r-k}(q)$.
Using (3.74), we intend to define the $(m-1)$-dimensional configuration space $\hat{M}_c \subset M_c$. The relation (3.74) is a nonlinear velocity constraint, which in general is a nonholonomic constraint. Nevertheless, because of this constraint imposed on the system, the equations which describe the dynamics are
$Q_2(q, \dot{q}, \ddot{q}) = 0, \ldots, Q_m(q, \dot{q}, \ddot{q}) = 0.$
Taking $\hat{q}^1 := q^2, \ldots, \hat{q}^{m-1} := q^m$, a coordinate system on the reduced configuration space $\hat{M}_c$, the Birkhoffian has in this case the form $\hat{\omega}_c = \sum_{j=1}^{m-1} \hat{Q}_j \mathrm{d}\hat{q}^j$,

$$\begin{aligned} \hat{Q}_j(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}}) = {} & \sum_{i=1}^{m-1} \sum_{a=r+1}^{r+k} \mathcal{N}_{(j+1)}^{a} \mathcal{N}_{(i+1)}^{a} L_{a-r} \left(\sum_{l=1}^{m-1} \mathcal{N}_{(j+1)}^{a} \dot{\hat{q}}^l \right) \ddot{\hat{q}}^i \\ & + \sum_{\Gamma=1}^{r} \mathcal{N}_{(j+1)}^{\Gamma} R_{\Gamma} \left(\mathcal{N}_1^{\Gamma} h(\dot{\hat{q}}^1, \ldots, \dot{\hat{q}}^m) + \sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^{\Gamma} \dot{\hat{q}}^l \right) \\ & + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_{(j+1)}^{\alpha} C_{\alpha-r-k} \left(\sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^{\alpha} \hat{q}^l \right) \end{aligned} \tag{3.75}$$

where $h : U \subset \mathbf{R}^{m-1} \longrightarrow \mathbf{R}$ is the unique function such that $h(\dot{\hat{q}}_0) = \dot{q}_0^1$, $\dot{q}_0^1 \in \mathbf{R}$, and

$$\sum_{\Gamma=1}^{r} \mathcal{N}_1^{\Gamma} R_{\Gamma} \left(\mathcal{N}_1^{\Gamma} h(\dot{\hat{q}}^1, \ldots, \dot{\hat{q}}^{m-1}) + \sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^{\Gamma} \dot{\hat{q}}^l \right) = 0 \tag{3.76}$$

for all $\dot{\hat{q}} = (\dot{\hat{q}}^1, \ldots, \dot{\hat{q}}^{m-1}) \in U$, with U a neighborhood of $\dot{\hat{q}}_0 = (\dot{\hat{q}}_0^1, \ldots, \dot{\hat{q}}_0^{m-1})$. One can prove, using the same ideas as in the previous case, that the Birkhoffian given by (3.75) is still dissipative.
If the determinant $\det\left[\sum_{a=r+1}^{r+k} \mathcal{N}_{(j+1)}^{a} \mathcal{N}_{(i+1)}^{a} \widetilde{L}_{a-r}\left(\dot{\hat{q}}\right)\right]_{i,j=1,\ldots,m-1} \neq 0$, then the Birkhoffian (3.75) is regular. □

If there exist in the network $m_3 < m$ loops which contain only resistors and capacitors, all the other loops containing at least an inductor, we can regularize *the Birkhoffian* (3.22) *by introducing into each of these m_3 loops an inductor in series, with the inductance functions $\mathcal{L}_{a'} : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$, $a' = 1, \ldots, m_3$, having very small values. The configuration space remains M_c of dimension m. Under certain conditions on the functions $\mathcal{L}_{a'}$ and on the functions L_a, $a = 1, \ldots, k$, which characterize the other inductors, the Birkhoffian ω_c^{ext} on M_c will be a* dissipative regular Birkhoffian.

Without loss of generality, we may assume that we have one loop in the network containing only resistors and capacitors and in the chosen coordinate system

$$\mathcal{N}_1^a = 0, \quad a = r+1, \ldots, r+k. \tag{3.77}$$

The expressions (3.22) of the Birkhoffian components become, $j = 2, \ldots, m$,

$$Q_1(q, \dot{q}, \ddot{q}) = \sum_{\Gamma=1}^{r} \mathcal{N}_1^\Gamma \widetilde{R}_\Gamma(\dot{q}) + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}_{\alpha-r-k}(q)$$

$$\begin{aligned} Q_j(q, \dot{q}, \ddot{q}) \ &= \sum_{i=2}^{m} \sum_{a=r+1}^{r+k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_{a-r}(\dot{q})\, \ddot{q}^i \\ &\quad + \sum_{\Gamma=1}^{r} \mathcal{N}_j^\Gamma \widetilde{R}_\Gamma(\dot{q}) + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^\alpha \widetilde{C}_{\alpha-r-k}(q). \end{aligned} \tag{3.78}$$

We introduce into this loop an inductor in series, described by the following relation between the current and the voltage

$$v = \mathcal{L}_1(\mathrm{I}) \frac{\mathrm{dI}}{\mathrm{d}t} \tag{3.79}$$

where $\mathcal{L}_1 : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ is a smooth invertible function. After introducing the inductor, the number of branches of the graph associated with the circuit increases by one, that is, there will be $b+1$ branches, and the number of nodes increases by one as well, that is, n becomes $n+1$. Still the cardinality of a selection of loops which cover the whole graph remains m. The configuration space is the same M_c, with dimension m. The corresponding Birkhoffian, denoted by ω_c^{ext}, has the component $Q_1(q, \dot{q}, \ddot{q})$ given by,

$$Q_1(q, \dot{q}, \ddot{q}) = \widetilde{\mathcal{L}}(\dot{q}^1)\ddot{q}^1 + \sum_{\Gamma=1}^{r} \mathcal{N}_1^\Gamma \widetilde{R}_\Gamma(\dot{q}) + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}_{\alpha-r-k}(q) \tag{3.80}$$

and the others $Q_2(q, \dot{q}, \ddot{q}), \ldots, Q_m(q, \dot{q}, \ddot{q})$ have the same form as in (3.78). This Birkhoffian is dissipative, the expression of the function E_{0_ω} on TM_c is (3.28) plus the term $\int \widetilde{\mathcal{L}}(\dot{q}^1)\dot{q}^1 \mathrm{d}\dot{q}^1$. The dissipative 1-form has the form (3.26).
If $\widetilde{\mathcal{L}}(\dot{q}^1) \det \left[\sum_{a=r+1}^{r+k} \mathcal{N}_{(j+1)}^a \mathcal{N}_{(i+1)}^a \widetilde{L}_{a-r}\left(\dot{\hat{q}}\right)\right]_{i,j=1,\ldots,m-1} \neq 0$, this Birkhoffian is regular. □

3.2. Voltage controlled resistors

Let us now consider the nonlinear resistors for which the constitutive relations are given by

$$\mathrm{I}_\Gamma = \mathfrak{R}_\Gamma(v_\Gamma), \quad \Gamma = 1, \ldots, r \tag{3.81}$$

where $\mathfrak{R}_\Gamma : \mathbf{R} \longrightarrow \mathbf{R}$ are smooth functions. In order to obtain a dissipative Birkhoffian, we also assume that, for all $x \neq 0$,

$$\mathfrak{R}_\Gamma(x)x > 0, \quad \forall \Gamma = 1, \ldots, r \tag{3.82}$$

that is, for each nonlinear resistor, the graph of the function $\mathfrak{R}_\Gamma$ lies in the union of the first and the third quadrant.
Taking into account (3.3)–(3.5) and (3.81), the Eqs. (3.1) and (3.2) governing the circuit have the form

$$\begin{cases} B^{\mathrm{T}}\begin{pmatrix} \mathrm{I}_\Gamma \\ \mathrm{I}_a \\ \dot{\mathrm{Q}}_\alpha \end{pmatrix} = 0 \\ \begin{cases} A^{\mathrm{T}}\begin{pmatrix} v_\Gamma \\ L_a(\mathrm{I}_a)\,\dot{\mathrm{I}}_a \\ C_\alpha(\mathrm{Q}_\alpha) \end{pmatrix} = 0 \\ \mathrm{I}_\Gamma = \mathfrak{R}_\Gamma(v_\Gamma). \end{cases} \end{cases} \tag{3.83}$$

Using the first set of equations (3.83), we define a family of m-dimensional affine–linear configuration spaces $M_c \subset \mathbf{R}^b$ parameterized by a constant vector c in $\mathbf{R}^n$. This vector is related to the initial values of the Q*-variables at some instant of time. A Birkhoffian ω_c on the configuration space M_c arises from the second set of equations (3.83). Thus, (M_c, ω_c) will be a family of Birkhoff systems that describe the RLC network considered.*

The first set of equations (3.83) is the same as the first set of equations (3.9). Thus, using them we can define the family M_c of m-dimensional affine–linear configuration spaces (3.11). For a coordinate system $q = (q^1, \ldots, q^m)$ on M_c, the relations between the x-coordinates (3.14) and the q-coordinates are given by (3.15). The matrix of constants $\mathcal{N}$ satisfies (3.16). Taking into account (3.20), we define the Birkhoffian ω_c of M_c such that the differential system (2.5) is the linear combination of the second set of equations in (3.83) obtained by replacing A^{T} with the matrix $\mathcal{N}^{\mathrm{T}}$. In terms of the q-coordinates chosen before, the components $Q_j(q, \dot{q}, \ddot{q})$ of the Birkhoffian have the implicit form

$$\begin{cases} Q_j(q,\dot{q},\ddot{q}) = F_j(\dot{q})\ddot{q} + G_j(q) + \displaystyle\sum_{\Gamma=1}^{r} \mathcal{N}_j^\Gamma v_\Gamma, \quad j = 1,\ldots,m \\ \displaystyle\sum_{j=1}^{m} \mathcal{N}_j^\Gamma \dot{q}^j = \mathfrak{R}_\Gamma(v_\Gamma) \end{cases} \tag{3.84}$$

where the functions $F_j(\dot{q})\ddot{q}$, $G_j(q)$ are given by (3.23) and (3.25).

The Birkhoffian (3.84) is dissipative.

Indeed, with the function $E_{0_\omega}(q, \dot{q})$ given by (3.28), the identity (2.16) becomes for the Birkhoffian (3.84),

$$\sum_{j=1}^{m} Q_j(q,\dot{q},\ddot{q})\dot{q}^j = \sum_{j=1}^{m}\left[\frac{\partial E_{0_\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j + \sum_{\Gamma=1}^{r}\mathcal{N}_j^\Gamma v_\Gamma \dot{q}^j\right]. \tag{3.85}$$

It remains to show that the vertical 1-form defined implicitly by

$$\begin{cases} D_j(q,\dot{q}) = \displaystyle\sum_{\Gamma=1}^{r} \mathcal{N}_j^\Gamma v_\Gamma, \quad j = 1,\ldots,m \\ \displaystyle\sum_{j=1}^{m} \mathcal{N}_j^\Gamma \dot{q}^j = \mathfrak{R}_\Gamma(v_\Gamma) \end{cases} \tag{3.86}$$

is dissipative. From the second set of relations in (3.86), we have

$$\sum_{j=1}^{m}\sum_{\Gamma=1}^{r}\mathcal{N}_j^{\Gamma} v_{\Gamma}\dot{q}^j=\sum_{\Gamma=1}^{r}\mathfrak{R}_{\Gamma}(v_{\Gamma})v_{\Gamma}. \tag{3.87}$$

Therefore, the inequality (2.14) reads $\sum_{\Gamma=1}^{r}\mathfrak{R}_{\Gamma}(v_{\Gamma})v_{\Gamma}>0$. The last inequality is satisfied in view of the assumption (3.82). □

As in the case of current controlled sources, the Birkhoffian (3.84) *is* never regular *if the network contains closed loops formed only by capacitors, or resistors, or both.*

4. Examples

Example 1. This example is based on the oriented connected graph:

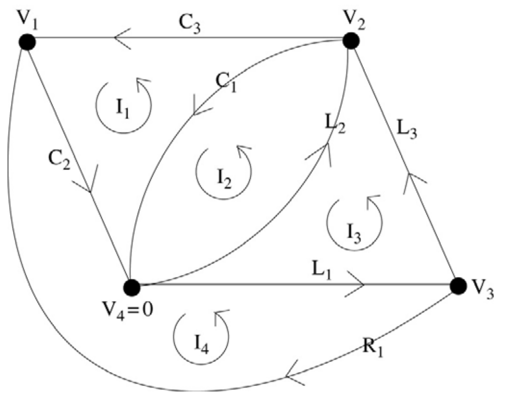

Fig. 1.

We have $r=1$, $k=3$, $p=3$, $n=3$, $m=4$, $b=7$. We choose the reference node to be V_4 and the current directions as indicated in Fig. 1. We cover the associated graph with the loops I_1, I_2, I_3, I_4. Let $V=(V_1,V_2,V_3)\in\mathbf{R}^3$ be the vector of node voltage values, $\mathrm{I}=(\mathrm{I}_{[\Gamma]},\mathrm{I}_{(a)},\mathrm{I}_{\alpha})\in\mathbf{R}^1\times\mathbf{R}^3\times\mathbf{R}^3$ be the vector of branch current values and $v=(v_{[\Gamma]},v_{(a)},v_{\alpha})\in\mathbf{R}^1\times\mathbf{R}^3\times\mathbf{R}^3$ be the vector of branch voltage values.
The branches in Fig. 1 are labelled as follows: the first branch is the resistive branch R_1, the second, the third and the fourth branch are the inductive branches L_1, L_2, L_3 and the last three branches are the capacitor branches C_1, C_2, C_3. The incidence and loop matrices, $B\in\mathfrak{M}_{73}(\mathbf{R})$ and $A\in\mathfrak{M}_{74}(\mathbf{R})$, can be written as

$$B=\begin{pmatrix}-1&0&1\\0&0&-1\\0&-1&0\\0&-1&1\\0&1&0\\1&0&0\\-1&1&0\end{pmatrix},\quad A=\begin{pmatrix}0&0&0&-1\\0&0&1&-1\\0&1&-1&0\\0&0&1&0\\-1&1&0&0\\1&0&0&-1\\1&0&0&0\end{pmatrix}. \tag{4.1}$$

One has $\mathrm{rank}(B)=3$, $\mathrm{rank}(A)=4$.

All the electrical devices are considered to be nonlinear and described by the relations (3.3) and (3.5), with $C_1, C_2, C_3 : \mathbf{R} \to \mathbf{R}\backslash\{0\}$, $L_1, L_2, L_3 : \mathbf{R} \to \mathbf{R}\backslash\{0\}$ smooth invertible functions, and by the relation (3.6), with $R_1 : \mathbf{R} \to \mathbf{R}$ a smooth function, such that (3.7) is satisfied, that is, for any $x \neq 0$

$$R_1(x)x > 0. \tag{4.2}$$

The equations (3.9) which govern the network have the form

$$\begin{cases} -\mathrm{I}_{[1]} + \dot{\mathrm{Q}}_2 - \dot{\mathrm{Q}}_3 = 0 \\ -\mathrm{I}_{(2)} - \mathrm{I}_{(3)} + \dot{\mathrm{Q}}_1 + \dot{\mathrm{Q}}_3 = 0 \\ \mathrm{I}_{[1]} - \mathrm{I}_{(1)} + \mathrm{I}_{(3)} = 0 \\ -C_1(\mathrm{Q}_1) + C_2(\mathrm{Q}_2) + C_3(\mathrm{Q}_3) = 0 \\ L_2(\mathrm{I}_{(2)})\dot{\mathrm{I}}_{(2)} + C_1(\mathrm{Q}_1) = 0 \\ L_1(\mathrm{I}_{(1)})\dot{\mathrm{I}}_{(1)} - L_2(\mathrm{I}_{(2)})\dot{\mathrm{I}}_{(2)} + L_3(\mathrm{I}_{(3)})\dot{\mathrm{I}}_{(3)} = 0 \\ -R_1(\mathrm{I}_{[1]}) - L_1(\mathrm{I}_{(1)})\dot{\mathrm{I}}_{(1)} - C_2(\mathrm{Q}_2) = 0. \end{cases} \tag{4.3}$$

The relations (3.12) and (3.14) read as follows for this example

$$\mathrm{I}_{[1]} := \dot{\mathrm{Q}}_{[1]}, \quad \mathrm{I}_{(a)} := \dot{\mathrm{Q}}_{(a)}, \quad a = 1, 2, 3 \tag{4.4}$$

$$x^1 := \mathrm{Q}_{[1]}, x^2 := \mathrm{Q}_{(1)}, x^3 := \mathrm{Q}_{(2)}, x^4 := \mathrm{Q}_{(3)}, x^5 := \mathrm{Q}_1, x^6 := \mathrm{Q}_2, x^7 := \mathrm{Q}_3. \tag{4.5}$$

Using the first three equations of the system (4.3) we define the 4-dimensional affine–linear configuration space M_c. In view of the notations (4.4), (4.5), we integrate these three equations and solving them in terms of 4 variables, we obtain, for example, $x^2 = x^1 + x^4 + \text{const}$, $x^5 = x^3 + x^4 - x^7 + \text{const}$, $x^6 = x^1 + x^7 + \text{const}$. Thus, a coordinate system on M_c is given by

$$q^1 := x^7, \quad q^2 := x^4, \quad q^3 := x^1, \quad q^4 := x^3. \tag{4.6}$$

The matrix of constants

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_j^{\Gamma} \\ \mathcal{N}_j^{a} \\ \mathcal{N}_j^{\alpha} \end{pmatrix}_{\substack{\Gamma=1,2,a=3,4,\alpha=5,6,7 \\ j=1,2,3,4}}$$

from (3.15) is given by

$$\mathcal{N} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, in terms of the q-coordinates (4.6), we may define the Birkhoffian $\omega_c = Q_1(q, \dot{q}, \ddot{q})\mathrm{d}q^1 + Q_2(q, \dot{q}, \ddot{q})\mathrm{d}q^2 + Q_3(q, \dot{q}, \ddot{q})\mathrm{d}q^3 + Q_4(q, \dot{q}, \ddot{q})\mathrm{d}q^4$ of M_c as in (3.22), with (3.23)–(3.25), that is,

$$\begin{aligned}
Q_1(q,\dot{q},\ddot{q}) &= -C_1(-q^1+q^2+q^4+\text{const})+C_2(q^1+q^3+\text{const})+C_3(q^1)\\
Q_2(q,\dot{q},\ddot{q}) &= \left[L_1(\dot{q}^2+\dot{q}^3)+L_3(\dot{q}^2)\right]\ddot{q}^2+L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^3\\
&\quad +C_1(-q^1+q^2+q^4+\text{const})\\
Q_3(q,\dot{q},\ddot{q}) &= L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^2+L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^3+R_1(\dot{q}^3)+C_2(q^1+q^3+\text{const})\\
Q_4(q,\dot{q},\ddot{q}) &= L_2(\dot{q}^4)\ddot{q}^4+C_1(-q^1+q^2+q^4+\text{const}).
\end{aligned} \tag{4.7}$$

The Birkhoffian (4.7) is *dissipative* and *not regular*.
Indeed, there exists a smooth function $E_{0_\omega}: TM \longrightarrow \mathbf{R}$ of the form (3.28), that is,

$$\begin{aligned}
E_{0_\omega}(q,\dot{q}) = &\int \widetilde{L}_1(\dot{q}^2,\dot{q}^3)(\dot{q}^2+\dot{q}^3)(\mathrm{d}\dot{q}^2+\mathrm{d}\dot{q}^3)+\int L_2(\dot{q}^4)\dot{q}^4\mathrm{d}\dot{q}^4+\int L_3(\dot{q}^2)\dot{q}^2\mathrm{d}\dot{q}^2\\
&-\int\int \widetilde{L}_1'(\dot{q}^2,\dot{q}^3)(\dot{q}^2+\dot{q}^3)\mathrm{d}\dot{q}^2\mathrm{d}\dot{q}^3-\int\int \widetilde{L}_1(\dot{q}^2,\dot{q}^3)\mathrm{d}\dot{q}^2\mathrm{d}\dot{q}^3\\
&+\int \widetilde{C}_1(q^1,q^2,q^4)(\mathrm{d}q^1-\mathrm{d}q^2+\mathrm{d}q^4)+\int \widetilde{C}_2(q^1,q^3)(\mathrm{d}q^1+\mathrm{d}q^3)\\
&+\int C_3(q^1)\mathrm{d}q^1-\int\int \widetilde{C}_1'(q^1,q^2,q^4)(-\mathrm{d}q^1\mathrm{d}q^2+\mathrm{d}q^1\mathrm{d}q^4-\mathrm{d}q^2\mathrm{d}q^4)\\
&-\int\int \widetilde{C}_2'(q^1,q^3)\mathrm{d}q^1\mathrm{d}q^3-\int\int\int \widetilde{C}_1''(q^1,q^2,q^4)\mathrm{d}q^1\mathrm{d}q^2\mathrm{d}q^4
\end{aligned} \tag{4.8}$$

such that (2.16) is satisfied with

$$D = R_1(\dot{q}^3)\mathrm{d}q^3. \tag{4.9}$$

Because the function R_1 satisfies (4.2), we obtain

$$D_j(q,\dot{q})\dot{q}^j = R_1(\dot{q}^3)\dot{q}^3 > 0 \tag{4.10}$$

that is, (4.9) is indeed a dissipative vertical 1-form.
We are in the case where the circuit considered has one loop which contains only capacitors. We note that for the Birkhoffian (4.7), the first row of the matrix $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2,3,4}$ contains only zeros. Therefore, $\det\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2,3,4} = 0$ and the Birkhoffian (4.7) is not regular.
Using the first relation in (4.7), we now define a three-dimensional $\bar{M}_c \subset M_c$ by

$$\begin{aligned}
\bar{M}_c = \Big\{ &q=(q^1,q^2,q^3,q^4)\in M_c/\, C_1(q^1-q^2+q^4+\text{const})\\
&+\; C_2(q^1+q^3+\text{const})+C_3(q^1)=0\Big\}.
\end{aligned} \tag{4.11}$$

By the implicit function theorem, we obtain a local coordinate system on the reduced configuration space $\bar{M}_c$. Taking $\bar{q}^1 := q^2, \bar{q}^2 := q^3, \bar{q}^3 := q^4$, the Birkhoffian has the form $\bar{\omega}_c = \sum_{j=1}^{3} \bar{Q}_j \mathrm{d}\bar{q}^j$, where

$$\begin{aligned}
\bar{Q}_1(\bar{q},\dot{\bar{q}},\ddot{\bar{q}}) &= \left[L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2)+L_3(\dot{\bar{q}}^1)\right]\ddot{\bar{q}}^1+L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2)\ddot{\bar{q}}^2\\
&\quad -C_1(f(\bar{q}^1,\bar{q}^2,\bar{q}^3)-\bar{q}^1+\bar{q}^3+\text{const})
\end{aligned}$$

$$\begin{aligned}\bar{Q}_2(\bar{q},\dot{\bar{q}},\ddot{\bar{q}}) &= L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2)\ddot{\bar{q}}^1 + L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2)\ddot{\bar{q}}^2 + R_1(\dot{\bar{q}}^2)\\ &\quad + C_2(f(\bar{q}^1,\bar{q}^2,\bar{q}^3)+\bar{q}^2+\text{const})\\ \bar{Q}_3(\bar{q},\dot{\bar{q}},\ddot{\bar{q}}) &= L_2(\dot{\bar{q}}^3)\ddot{\bar{q}}^3 + C_1(f(\bar{q}^1,\bar{q}^2,\bar{q}^3)-\bar{q}^1+\bar{q}^3+\text{const})\end{aligned} \tag{4.12}$$

$f : U \subset \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ being the unique function such that $f(\bar{q}_0) = q_0^1$, $q_0^1 \in \mathbf{R}$, and $C_1(f(\bar{q})-\bar{q}^1+\bar{q}^3+\text{const})+C_2(f(\bar{q})+\bar{q}^2+\text{const})+C_3(f(\bar{q}))=0$, $\forall \bar{q}=(\bar{q}^1,\bar{q}^2,\bar{q}^3)\in U$, with U a neighborhood of $\bar{q}_0=(\bar{q}_0^1,\bar{q}_0^2,\bar{q}_0^3)$.

We have shown in Section 3 that in this case the reduced Birkhoffian (4.12) is *dissipative* and *regular*.

The relations (3.45) are satisfied for this example, thus, there exists a function $\bar{E}_{0_\omega}(\bar{q},\dot{\bar{q}})$ such that (3.41) is fulfilled, with the dissipative 1-form given by

$$D = R_1(\dot{\bar{q}}^2)\mathrm{d}\bar{q}^2. \tag{4.13}$$

We calculate

$$\det\left[\frac{\partial \bar{Q}_j}{\partial \ddot{\bar{q}}^i}\right]_{i,j=1,2,3} = \begin{vmatrix} L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2)+L_3(\dot{\bar{q}}^1) & L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2) & 0\\ L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2) & L_1(\dot{\bar{q}}^1+\dot{\bar{q}}^2) & 0\\ 0 & 0 & L_2(\dot{\bar{q}}^3)\end{vmatrix}. \tag{4.14}$$

Because $L_1, L_2, L_3 : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$, the determinant above is different from zero, then, the reduced Birkhoffian given by (4.12) is regular. □

If the nonlinear resistor is voltage controlled, that is,

$$\mathrm{I}_{[1]} = \mathfrak{R}_1(v_1) \tag{4.15}$$

we obtain, instead of (4.7), the following implicit Birkhoffian

$$\begin{cases} Q_1(q,\dot{q},\ddot{q}) = C_1(q^1-q^2+q^4+\text{const})+C_2(q^1+q^3+\text{const})+C_3(q^1)\\ Q_2(q,\dot{q},\ddot{q}) = \left[L_1(\dot{q}^2+\dot{q}^3)+L_3(\dot{q}^2)\right]\ddot{q}^2+L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^3\\ \qquad\qquad\quad -C_1(q^1-q^2+q^4+\text{const})\\ Q_3(q,\dot{q},\ddot{q}) = L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^2+L_1(\dot{q}^2+\dot{q}^3)\ddot{q}^3+C_2(q^1+q^3+\text{const})+v_1\\ Q_4(q,\dot{q},\ddot{q}) = L_2(\dot{q}^4)\ddot{q}^4+C_1(q^1-q^2+q^4+\text{const})\\ \dot{q}^3 = \mathfrak{R}_1(v_1).\end{cases} \tag{4.16}$$

We suppose that the nonlinear voltage controlled resistor satisfies $\mathfrak{R}_1(x)x > 0$, for any $x \neq 0$. Thus, as in the case of current controlled resistors, there exists a function $E_{0_\omega}(q,\dot{q})$ given by (4.8), such that (2.16) is satisfied with the dissipative implicit 1-form

$$\begin{cases} D(q,\dot{q}) = v_1\mathrm{d}q^3,\\ \dot{q}^3 = \mathfrak{R}_1(v_1).\end{cases} \tag{4.17}$$

Therefore, the Birkhoffian (4.16) is *dissipative*. The Birkhoffian (4.16) is also *not regular*. □

Example 2. This example is based on the oriented connected graph:

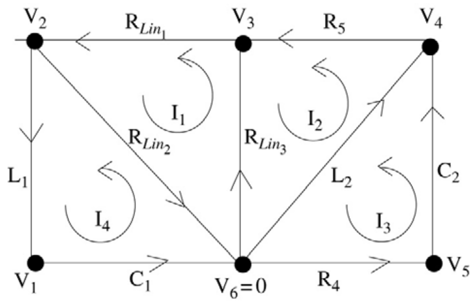

Fig. 2.

We have $r = 5$, $k = 2$, $p = 2$, $n = 5$, $m = 4$, $b = 9$. We choose the reference node to be V_6 and the current directions as indicated in Fig. 2. We cover the associated graph with the loops I_1, I_2, I_3, I_4. Let $V = (V_1, V_2, V_3, V_4, V_5) \in \mathbf{R}^5$ be the vector of node voltage values, $\mathrm{I} = (\mathrm{I}_{[\Gamma]}, \mathrm{I}_{(a)}, \mathrm{I}_\alpha) \in \mathbf{R}^5 \times \mathbf{R}^2 \times \mathbf{R}^2$ be the vector of branch current values and $v = (v_{[\Gamma]}, v_{(a)}, v_\alpha) \in \mathbf{R}^5 \times \mathbf{R}^2 \times \mathbf{R}^2$ be the vector of branch voltage values.
The branches in Fig. 2 are labelled as follows: the first, the second and the third branch are the linear resistive branches, $\mathrm{R}_{\mathrm{lin}_1}, \mathrm{R}_{\mathrm{lin}_2}, \mathrm{R}_{\mathrm{lin}_3}$, the fourth and the fifth branch are the nonlinear resistive branches R_4, R_5, the sixth and the seventh branch are the inductive branches L_1, L_2, and the last two branches are the capacitor branches C_1, C_2. The incidence and loop matrices, $B \in \mathfrak{M}_{95}(\mathbf{R})$ and $A \in \mathfrak{M}_{94}(\mathbf{R})$, can be written as

$$B = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.18}$$

One has $\mathrm{rank}(B) = 5$, $\mathrm{rank}(A) = 4$.
Except for the resistors in the first loop which are considered linear, all devices are nonlinear and are described by the relation (3.3), (3.5) and (3.6). We suppose that $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3 > 0$ are distinct constants, $C_1, C_2 : \mathbf{R} \to \mathbf{R}\backslash\{0\}$, $L_1, L_2 : \mathbf{R} \to \mathbf{R}\backslash\{0\}$, smooth invertible functions and $R_4, R_5 : \mathbf{R} \to \mathbf{R}$ smooth functions such that, for any $x \neq 0$

$$R_4(x)x > 0, \quad R_5(x)x > 0. \tag{4.19}$$

The first set of equations (3.9) has the form

$$\begin{cases} -\mathrm{I}_{(1)} + \dot{\mathrm{Q}}_1 = 0 \\ -\mathrm{I}_{[1]} + \mathrm{I}_{[2]} + \mathrm{I}_{(1)} = 0 \\ \mathrm{I}_{[1]} - \mathrm{I}_{[3]} - \mathrm{I}_{[5]} = 0 \\ \mathrm{I}_{[5]} + \mathrm{I}_{(2)} - \dot{\mathrm{Q}}_2 = 0 \\ -\mathrm{I}_{[4]} + \dot{\mathrm{Q}}_2 = 0. \end{cases} \tag{4.20}$$

The relations (3.12) and (3.14) read as follows for this example

$$\mathrm{I}_{[\Gamma]} := \dot{\mathrm{Q}}_{[\Gamma]}, \quad \Gamma = 1, \ldots, 5, \quad \mathrm{I}_{(a)} := \dot{\mathrm{Q}}_{(a)}, \quad a = 1, 2 \tag{4.21}$$

$$x^1 := \mathrm{Q}_{[1]}, \ldots, x^5 := \mathrm{Q}_{[5]}, x^6 := \mathrm{Q}_{(1)}, x^7 := \mathrm{Q}_{(2)}, x^8 := \mathrm{Q}_1, x^9 := \mathrm{Q}_2. \tag{4.22}$$

Using the equations from (4.20), we define the four-dimensional affine–linear configuration space M_c. In view of the notation (4.21) and (4.22), we integrate these five equations and solving them in terms of four variables, we obtain, for example, $x^2 = x^1 - x^8 + \text{const}$, $x^3 = x^1 - x^5 + \text{const}$, $x^4 = x^9 + \text{const}$, $x^6 = x^8 + \text{const}$, $x^7 = -x^5 + x^9 + \text{const}$. Thus, a coordinate system on M_c is given by

$$q^1 := x^1, \quad q^2 := x^5, \quad q^3 := x^8, \quad q^4 := x^9 \tag{4.23}$$

The matrix of constants $\mathcal{N}$ from (3.15) is given by

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, in terms of the q-coordinates (4.23), we define the Birkhoffian $\omega_c = Q_1(q, \dot{q}, \ddot{q})\mathrm{d}q^1 + Q_2(q, \dot{q}, \ddot{q})\mathrm{d}q^2 + Q_3(q, \dot{q}, \ddot{q})\mathrm{d}q^3 + Q_4(q, \dot{q}, \ddot{q})\mathrm{d}q^4$ of M_c as in (3.22), with (3.23)–(3.25), that is,

$$\begin{aligned} Q_1(q, \dot{q}, \ddot{q}) &= (\mathrm{R}_1 + \mathrm{R}_2 + \mathrm{R}_3)\dot{q}^1 - \mathrm{R}_3\dot{q}^2 - \mathrm{R}_2\dot{q}^3 \\ Q_2(q, \dot{q}, \ddot{q}) &= L_2(-\dot{q}^2 + \dot{q}^4)\ddot{q}^2 - L_2(-\dot{q}^2 + \dot{q}^4)\ddot{q}^4 - \mathrm{R}_3\dot{q}^1 + \mathrm{R}_3\dot{q}^2 + R_5(\dot{q}^2) \\ Q_3(q, \dot{q}, \ddot{q}) &= L_1(\dot{q}^3)\ddot{q}^3 - \mathrm{R}_2\dot{q}^1 + \mathrm{R}_2\dot{q}^3 + C_1(q^3) \\ Q_4(q, \dot{q}, \ddot{q}) &= -L_2(-\dot{q}^2 + \dot{q}^4)\ddot{q}^2 + L_2(-\dot{q}^2 + \dot{q}^4)\ddot{q}^4 + R_4(\dot{q}^4) + C_2(q^4). \end{aligned} \tag{4.24}$$

The Birkhoffian (4.24) is *dissipative* and *not regular*.
Indeed, there exists a smooth function $E_{0_\omega} : TM \to \mathbf{R}$ of the form (3.28), that is,

$$\begin{aligned} E_{0_\omega}(q, \dot{q}) = &\int L_1(\dot{q}^3)\dot{q}^3\mathrm{d}\dot{q}^3 + \int \widetilde{L}_2(\dot{q}^2, \dot{q}^4)(-\dot{q}^2 + \dot{q}^4)(-\mathrm{d}\dot{q}^2 + \mathrm{d}\dot{q}^4) \\ &- \int\int \widetilde{L}_2'(\dot{q}^2, \dot{q}^4)(-\dot{q}^2 + \dot{q}^4)\mathrm{d}\dot{q}^2\mathrm{d}\dot{q}^4 + \int\int \widetilde{L}_2(\dot{q}^2, \dot{q}^4)\mathrm{d}\dot{q}^2\mathrm{d}\dot{q}^4 \\ &+ \int C_1(q^3)\mathrm{d}q^3 + \int C_2(q^4)\mathrm{d}q^4 \end{aligned} \tag{4.25}$$

such that (2.16) is satisfied with the dissipative 1-form defined by

$$\begin{aligned} D = &\left[\dot{q}^1\mathrm{R}_1 + (\dot{q}^1 - \dot{q}^3)\mathrm{R}_2 + (\dot{q}^1 - \dot{q}^2)\mathrm{R}_3\right]\mathrm{d}q^1 \\ &+ \left[-(\dot{q}^1 - \dot{q}^2)\,\mathrm{R}_3 + R_5(\dot{q}^2)\right]\mathrm{d}q^2 - (\dot{q}^1 - \dot{q}^3)\mathrm{R}_2\mathrm{d}q^3 + R_4(\dot{q}^4)\mathrm{d}q^4. \end{aligned} \tag{4.26}$$

In view of the assumptions (4.19) and of $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3 > 0$, we get

$$D_j(q,\dot{q})\dot{q}^j = \mathrm{R}_1(\dot{q}^1)^2 + \mathrm{R}_2(\dot{q}^1-\dot{q}^3)^2 + \mathrm{R}_3(\dot{q}^1-\dot{q}^2)^2 + R_4(\dot{q}^4)\dot{q}^4 + R_5(\dot{q}^2)\dot{q}^2 > 0. \tag{4.27}$$

Therefore, the vertical 1-form in (4.26) is dissipative.

We are in the case where the circuit considered has one loop which contains only resistors. We note that for the Birkhoffian (4.24), the first row of the matrix $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,\ldots,4}$ contains only zeros. Therefore, $\det\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,\ldots,4} = 0$ and the Birkhoffian (4.24) is not regular.

Using the first relation in (4.24), we now define $\hat{M}_c \subset M_c$ by

$$\hat{M}_c = \left\{ q=(q^1,q^2,q^3,q^4) \in M_c/(\mathrm{R}_1+\mathrm{R}_2+\mathrm{R}_3)q^1 - \mathrm{R}_3 q^2 - \mathrm{R}_2 q^3 + c_1 = 0 \right\} \tag{4.28}$$

where c_1 is a real constant.

On the reduced configuration space $\hat{M}_c$, in the coordinate system given by $\hat{q}^1 := q^2, \hat{q}^2 := q^3, \hat{q}^3 := q^4$, the Birkhoffian has the form $\hat{\omega}_c = \hat{Q}_1 \mathrm{d}\hat{q}^1 + \hat{Q}_2 \mathrm{d}\hat{q}^2 + \hat{Q}_3 \mathrm{d}\hat{q}^3$

$$\begin{aligned}
\hat{Q}_1(\hat{q},\dot{\hat{q}},\ddot{\hat{q}}) &= L_2(-\dot{\hat{q}}^1+\dot{\hat{q}}^3)\ddot{\hat{q}}^1 - L_2(-\dot{\hat{q}}^1+\dot{\hat{q}}^3)\ddot{\hat{q}}^3 + (\mathfrak{C}_2+\mathfrak{C}_3)\,\dot{\hat{q}}^1 - \mathfrak{C}_3\dot{\hat{q}}^2 + R_5(\dot{\hat{q}}^1)\\
\hat{Q}_2(\hat{q},\dot{\hat{q}},\ddot{\hat{q}}) &= L_1(\dot{\hat{q}}^2)\ddot{\hat{q}}^2 - \mathfrak{C}_3\dot{\hat{q}}^1 + (\mathfrak{C}_1+\mathfrak{C}_3)\,\dot{\hat{q}}^2 + C_1(\hat{q}^2)\\
\hat{Q}_3(\hat{q},\dot{\hat{q}},\ddot{\hat{q}}) &= -L_2(-\dot{\hat{q}}^1+\dot{\hat{q}}^3)\ddot{\hat{q}}^1 + L_2(-\dot{\hat{q}}^1+\dot{\hat{q}}^3)\ddot{\hat{q}}^3 + R_4(\dot{\hat{q}}^3) + C_2(\hat{q}^3)
\end{aligned} \tag{4.29}$$

where we denote the constants as $\mathfrak{C}_1 := \frac{\mathrm{R}_1\mathrm{R}_2}{\mathrm{R}_1+\mathrm{R}_2+\mathrm{R}_3}$, $\mathfrak{C}_2 := \frac{\mathrm{R}_1\mathrm{R}_3}{\mathrm{R}_1+\mathrm{R}_2+\mathrm{R}_3}$, $\mathfrak{C}_3 := \frac{\mathrm{R}_2\mathrm{R}_3}{\mathrm{R}_1+\mathrm{R}_2+\mathrm{R}_3}$.

As we have stated in Section 3, the Birkhoffian given by (4.29) is still *dissipative*. The function $\hat{E}_{0_\omega}(\hat{q},\dot{\hat{q}})$ has the same form (4.25) written in the coordinates $\hat{q}$. The relation (3.62) is satisfied with the dissipative 1-form defined by

$$\hat{D} = \left[\dot{\hat{q}}^1\mathfrak{C}_2 + (\dot{\hat{q}}^1-\dot{\hat{q}}^2)\mathfrak{C}_3 + R_5(\dot{\hat{q}}^1)\right]\mathrm{d}\hat{q}^1 + \left[\dot{\hat{q}}^2\mathfrak{C}_1 - (\dot{\hat{q}}^1-\dot{\hat{q}}^2)\mathfrak{C}_3\right]\mathrm{d}\hat{q}^2 + R_4(\dot{\hat{q}}^3)\mathrm{d}\hat{q}^3. \tag{4.30}$$

The vertical 1-form above is dissipative, as can be seen as follows: For $\mathrm{R}_1 > 0$, $\mathrm{R}_2 > 0$, $\mathrm{R}_3 > 0$, we get $\mathfrak{C}_1 > 0$, $\mathfrak{C}_2 > 0$, $\mathfrak{C}_3 > 0$ and together with (4.19) these yield

$$\hat{D}_j(\hat{q},\dot{\hat{q}})\dot{\hat{q}}^j = \mathfrak{C}_2(\dot{\hat{q}}^1)^2 + \mathfrak{C}_3(\dot{\hat{q}}^1-\dot{\hat{q}}^2)^2 + \mathfrak{C}_1(\dot{\hat{q}}^2)^2 + R_5(\dot{\hat{q}}^1)\dot{\hat{q}}^1 + R_4(\dot{\hat{q}}^3)\dot{\hat{q}}^3 > 0. \tag{4.31}$$

The Birkhoffian given by (4.29) is *not regular*, since the determinant

$$\det\left[\frac{\partial \hat{Q}_j}{\partial \ddot{\hat{q}}^i}\right]_{i,j=1,2,3} = \begin{vmatrix} \widetilde{L}_2(\dot{\hat{q}}) & 0 & -\widetilde{L}_2(\dot{\hat{q}}) \\ 0 & \widetilde{L}_1(\dot{\hat{q}}) & 0 \\ -\widetilde{L}_2(\dot{\hat{q}}) & 0 & \widetilde{L}_2(\dot{\hat{q}}) \end{vmatrix} = 0. \tag{4.32}$$

This result is not unexpected because the network considered also has a loop which contains only resistors and capacitors, formed by R_4, C_2, R_5, R_{lin_3}. In order to regularize the Birkhoffian (4.29), we introduce an inductor in series into this loop, described by the following relation between the current and the voltage: $v = \mathcal{L}_1(\mathrm{I})\frac{\mathrm{dI}}{\mathrm{d}t}$, $\mathcal{L}_1 : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ being a smooth invertible function. This means that this loop will have one more node and one more branch. The number of branches for the graph associated with the circuit increases by one, that is, there will be $b = 10$ branches, and the number of nodes increases by one as well, that is, we will have $n = 6$. But the cardinality of a selection of loops which cover the whole graph remains $m = 4$. After the calculation we arrive at the reduced configuration space defined by (4.28). On the reduced configuration space $\hat{M}_c$, in the coordinate system given by $\hat{q}^1 := q^2, \hat{q}^2 := q^3, \hat{q}^3 := q^4$, the Birkhoffian $\hat{\omega}_c^{\text{ext}}$ has the components $\hat{Q}_1(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}})$, $\hat{Q}_2(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}})$ given by (4.29) and the expression for $\hat{Q}_3(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}})$ becomes

$$\hat{Q}_3(\hat{q}, \dot{\hat{q}}, \ddot{\hat{q}}) = -L_2(-\dot{\hat{q}}^1 + \dot{\hat{q}}^3)\ddot{\hat{q}}^1 + \left[\mathcal{L}_1(\dot{\hat{q}}^3) + L_2(-\dot{\hat{q}}^1 + \dot{\hat{q}}^3)\right]\ddot{\hat{q}}^3 + R_4(\dot{\hat{q}}^3) + C_2(\hat{q}^3). \tag{4.33}$$

We now calculate

$$\det\left[\frac{\partial \hat{Q}_j}{\partial \ddot{\hat{q}}^i}\right]_{i,j=1,2,3} = \begin{vmatrix} \widetilde{L}_2(\dot{\hat{q}}) & 0 & -\widetilde{L}_2(\dot{\hat{q}}) \\ 0 & \widetilde{L}_1(\dot{\hat{q}}) & 0 \\ -\widetilde{L}_2(\dot{\hat{q}}) & 0 & \widetilde{\mathcal{L}}_1(\dot{\hat{q}}) + \widetilde{L}_2(\dot{\hat{q}}) \end{vmatrix}. \tag{4.34}$$

Because $\mathcal{L}_1, L_1, L_2 : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ the determinant above is different from zero; then, the Birkhoffian $\hat{\omega}_c^{\text{ext}}$ is regular. □

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